CHARACTER SHEAVES ON DISCONNECTED GROUPS, IV

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Introduction

Throughout this paper, G denotes a fixed. not necessarily connected, reductive algebraic group over an algebraically closed field $\mathbf{k}$. This paper is a part of a series (beginning with [L10], [L11], [L12]) which attempts to develop a theory of character sheaves on G. The numbering of the sections and references continues that of the earlier Parts.

Assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$ and that G has a fixed $\mathbf{F}_q$-rational structure with Frobenius map $F : G \to G$. To any triple $(L, S, \mathcal{E})$ (where L is a Levi of a parabolic of G^0, S is an isolated stratum of the normalizer of L, with certain properties, and $\mathcal{E}$ is an irreducible cuspidal local system on S) we have an associated in 5.6 a (not necessarily irreducible) intersection cohomology complex $\mathfrak{K}$ on G. If $F(L) = L, F(S) = S$ and we are given an isomorphism $F^*\mathcal{E} \to \mathcal{E}$, there is an induced isomorphism $\phi : F^*\mathfrak{K} \to \mathfrak{K}$ hence the characteristic function $\chi_{\mathfrak{K},\phi} : G^F \to \bar{\mathbf{Q}}_l$ is well defined.

The main result of this paper (Theorem 21.14) is that the functions $\chi_{\mathfrak{K},\phi}$ that are not identically zero (for various $(L, S, \mathcal{E})$ up to G^{0F}-conjugacy) form a $\bar{\mathbf{Q}}_l$-basis of the vector space $\mathbf{V}$ of functions $G^F \to \bar{\mathbf{Q}}_l$ that are constant on G^{0F}-conjugacy classes. The proof uses several of the results developed in earlier Parts (the generalized Springer correspondence in §11, the generalized Green functions in §15, the character formula in §16). It also uses the classification of cuspidal local systems (this is needed in §17 which is a preliminary to the proof of Theorem 21.14).

A corollary of the main theorem is Theorem 21.21 which states that the characteristic functions of admissible complexes A such that $F^*A \cong A$ form a basis for $\mathbf{V}$. In the connected case such a result was proved in [L13] subject to some mild restrictions on the characteristic. The present proof has no restrictions on the characteristic and it makes no use of the orthogonality formulas which will appear in a later stage of the theory.

Another corollary of the main theorem is the construction in §22 of a "twisted induction" map from certain functions on a subgroup of G^F to functions on G^F.

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Our paper contains also a new characterization of isolated elements (see 18.2) which is obvious in the connected case but less obvious in the disconnected case.

Notation. If Γ is a finite group with a given automorphism $F : \Gamma \xrightarrow{\sim} \Gamma$, the "$F$-twisted conjugacy classes" of Γ are the orbits of the Γ-action on Γ given by $y : w \mapsto F^{-1}(y)wy^{-1}$.

We shall denote by σ_G the map from G to the set of G^0-conjugacy classes of quasi-semisimple elements in G defined in 7.1 (where it is denoted by σ).

For two elements a, b of a group we set $a^b = b^{-1}ab$.

Let $p \geq 0$ be the characteristic of $\mathbf{k}$.

Errata for Part I.

In 1.11 replace [B, 9.8] by: A.Borel and J.Tits, *Groupes réductifs*, Publ.Math.IHES 27(1965), 55-150, Lemma 11.1.

6.7: line 3,4: replace $\tilde{\mathcal{E}}$ by $\pi_! \tilde{\mathcal{E}}$.

p.403 line 5: replace $Z_{L_1}(g_s)$ by $\dim Z_{L_1}(g_s)$.

Contents

17. Properties of cuspidal classes

17.1. This section contains the proof of a key property (17.13) of "cuspidal conjugacy classes (see below) which is needed in the proof of the main results of §21.

17.2. A G^0-conjugacy class in G is said to be *isolated* if one (or equivalently, any) element of it is isolated in G, see 2.2. We show:

(a) *if G^0 is semisimple and $g \in G$ is isolated then g_s has finite order.*

We may assume that g_u is quasi-semisimple in $Z_G(g_s)$. Let $H = Z_G(g_u)$. As in the proof of Lemma 2.7, we see that H^0 is semisimple. By Lemma 2.5, g_s is isolated in H. Replacing G by H, we are reduced to the case where g is semisimple. In this case we argue by induction on $\dim G$. Assume first that $\dim Z_G(g) < \dim G$. Now $Z_G(g)^0$ is semisimple and g is isolated in $Z_G(g)$. By the induction hypothesis g has finite order. Assume next that $\dim Z_G(g) = \dim G$ that is, $Z_G(g)^0 = G^0$. We can find an integer $n \geq 1$ such that $g^n \in G^0$. Clearly, $Z_G(g)^0 \subset Z_G(g^n)^0$. Hence $Z_G(g^n)^0 = G^0$. Thus, $g^n \in \mathcal{Z}_{G^0}$. Since $\mathcal{Z}_{G^0}$ is finite, we see that g^n has finite order. This proves (a).

17.3. Let $\mathbf{c}$ be an isolated G^0-conjugacy class in G and let $\mathcal{F}$ be a local system on $\mathbf{c}$. Let $[\mathcal{F}]$ be the isomorphism class of $\mathcal{F}$. We say that $(\mathbf{c},\mathcal{F})$ (or $(\mathbf{c},[\mathcal{F}])$ or $\mathcal{F}$) is *cuspidal* if $\mathcal{F}$ is G^0-equivariant and for any proper parabolic P of G^0 and any U_P-coset R in N_GP we have $H_c^d(\mathbf{c}\cap R,\mathcal{F})=0$ where d is $\dim \mathbf{c}$ minus the dimension of the P/U_P-conjugacy class of R/U_P in N_GP/U_P.

A G^0-conjugacy class in G is said to be *cuspidal* if it is isolated and if it carries some non-zero cuspidal local system.

Let $\mathbf{c}$ be an isolated G^0-conjugacy class in G and let $\mathcal{F}$ be a G^0-equivariant local system on $\mathbf{c}$. Then

(a) *$(\mathbf{c},\mathcal{F})$ is cuspidal if and only if for some/any $g\in\mathbf{c}$, and some/any unipotent $Z_G(g_s)^0$-conjugacy class $\mathfrak{c}$ of $Z_G(g_s)$ contained in $\{u\in Z_G(g_s); u \text{ unipotent}, g_su\in \mathbf{c}\}$, the local system $j^*\mathcal{F}$ on $\mathfrak{c}$ is cuspidal relative to $Z_G(g_s)$ (here $j:\mathfrak{c}\to\mathbf{c}$ is $u\mapsto g_su$).*

The proof of (a) is identical to that of its special case 6.6.

17.4. (a) *Let $g\in G$ be quasi-semisimple. Then $Z_{G^0}(g)/Z_G(g)^0$ is a diagonalizable group.*

(b) *If in addition, G^0 is semisimple, simply connected, we have $Z_{G^0}(g)=Z_G(g)^0$.*

(b) is proved in [St, 8.1]; in the closely related case of compact Lie groups, it goes back to the earlier paper [B1, 3.4]. In the case where G^0 is semisimple, (a) is proved in [St, 9.1] using (b). For the general case see [DM, 1.6(i)], [DM, 1.24].

Lemma 17.5. *Assume that G is such that G^0 is semisimple, simply connected. Let $s\in G$ be semisimple and $u\in G$ be unipotent such that $su=us$ and the G^0-conjugacy class of su is cuspidal. Let $z\in\mathcal{Z}_{G^0}$. Let $\underline{Z}=Z_G(s)^0$. Assume that $g\in G^0, g^s=zg, g^u=yg$ with $y\in\underline{Z}^0$. Then $y=y'^uy'^{-1}$ for some $y'\in\underline{Z}^0$.*

The proof (carried out in 17.6-17.11) consists of a number of steps which reduce us to the case where G^0 is almost simple, simply connected, in which case we shall use the classification of cuspidal conjugacy classes (which by 17.3(a) follows from the results on the classification of unipotent cuspidal conjugacy classes given in §12 and [L2]).

17.6. In the setup of 17.5 assume that

(a) *G^0 is almost simple and $G=G^0\times C_m$ (semidirect product, C_m a cyclic group of order $m\ge 1$ with generator e) with group structure $(a,e^t)(a',e^{t'})=(a\alpha^t(a'),e^{t+t'})$ where $\alpha:G^0\to G^0$ is an automorphism of order $m\ge 1$ preserving an épinglage and su is of the form (x,e) for some $x\in G^0$.*

If $\mathcal{Z}_{G^0}=\{1\}$ then we must have $z=1$ and we can take $y'=g$. This handles the following types of G^0:

B_n, C_n (with $m=1$, $p=2$), D_n (with $m\in\{1,2\}, p=2$)

D_4 (with $m=3, p=2$)

E_6 (with $m\in\{1,2\}, p=3$), E_7 (with $p=2$),

E_8, F_4 (with $m \in \{1,2\}$), G_2.

If $\underline{Z} = G^0$ then we must have $z = 1$ and we can take $y' = g$. This handles the following types of G^0:

A_n (with $m = 1$), A_n (with $m = 2, p = 2$),

E_6 (with $m = 1, p = 2$, $\underline{Z} = G^0$), E_7 (with $p = 3$, $\underline{Z} = G^0$).

Let $\mathbf{c}_1$ (resp. $\mathbf{c}_2$) be the $\underline{Z}^0$-conjugacy class of u (resp. uy) in $Z_G(s)$. Then $\mathrm{Ad}(g) : Z_G(s) \xrightarrow{\sim} Z_G(s)$ carries $\mathbf{c}_1$ to $\mathbf{c}_2$. Thus $\mathbf{c}_1, \mathbf{c}_2$ are two cuspidal unipotent $\underline{Z}$-conjugacy classes in the same connected component of $Z_G(s)$. It is enough to show that $\mathbf{c}_1 = \mathbf{c}_2$. (Then $uy = y'uy'^{-1}$ for some $y' \in \underline{Z}$.) If there is only one cuspidal unipotent $\underline{Z}$-conjugacy class in $u\underline{Z}$ then clearly $\mathbf{c}_1 = \mathbf{c}_2$. This handles the following types of G^0:

A_n (with $m = 2, p \neq 2$), C_n, D_4 (with $m = 3, p \neq 2$),

E_6 (with $m = 1, p \neq 2, 3$), E_6 (with $m = 1, p = 2, \underline{Z} \neq G^0$),

E_6 (with $m = 2, p \neq 3$),

E_7 (with $p \neq 2, 3$), E_7 (with $p = 3, \underline{Z} \neq G^0$).

The cases not covered by the arguments above are with G_0 of type B_n or D_n with $m \in \{1,2\}$ and $p \neq 2$. In each of these cases there are at most two cuspidal unipotent $\underline{Z}$-conjugacy classes in $Z_G(s)$ (they are automatically contained in $\underline{Z}$). Each of these classes is stable under any automorphism of $\underline{Z}$; in particular, under $\mathrm{Ad}(g)$. Hence these cases are settled. Thus, Lemma 17.5 holds in the present case.

17.7. In the setup of 17.5 assume that

(a) G^0 *is almost simple and* $G = G^0 \times C_n$ *(semidirect product,* C_n *a cyclic group of order* $n \geq 1$ *with generator* e*) with group structure* $(a, e^t)(a', e^{t'}) = (a\alpha^t(a'), e^{t+t'})$ *where* $\alpha : G^0 \to G^0$ *is an automorphism preserving an épinglage such that* $\alpha^n = 1$ *and* su *is of the form* (x, e) *for some* $x \in G^0$*.*

Let m be the order of α. Thus, $n/m \in \mathbf{Z}$. Let $\bar{G} = G^0 \times C_m$ (semidirect product, C_n a cyclic group of order $m \geq 1$ with generator e') with group structure $(a, e'^t)(a', e'^{t'}) = (a\alpha^t(a'), e'^{t+t'})$. Let $\pi : G \to \bar{G}$, $(g, e^t) \mapsto (g, e'^t)$. Then π induces $G^0 \xrightarrow{\sim} \bar{G}^0$ with kernel $K = \{1, e^m, e^{2m}, \dots\}$. From the definitions we see that the $\bar{G}^0$-conjugacy class of $\pi(s)\pi(u)$ is cuspidal. Applying 17.6 to $\bar{G}, \pi(s), \pi(u), \pi(g), \pi(z), \pi(y)$ instead of G, s, u, g, z, y, we see that there exists $y' \in G^0$ such that $sy' = y'sk$, $uy = y'uy'^{-1}k'$ with $k, k' \in K$. Applying the homomorphism $\rho : G \to C_n, (g, e^t) \mapsto e^t$ we get

$\rho(s)\rho(y') = \rho(y')\rho(s)\rho(k)$, $\rho(u)\rho(y) = \rho(y')\rho(u)\rho(y')^{-1}\rho(k')$,

$\rho(u)^{-1}\rho(g)\rho(u) = \rho(y)\rho(g)$.

Using the commutativity of C_n we deduce $\rho(y) = 1, \rho(k') = 1, \rho(k) = 1$. Since $\rho : K \to C_n$ is injective it follows that $k = k' = 1$. Thus, Lemma 17.5 holds in the present case.

17.8. In the setup of 17.5 assume that

(a) G^0 *is almost simple and* G *is generated by the connected component* D *that contains* su*.*

We can find $d \in D$ such that $\mathrm{Ad}(d) : G^0 \to G^0$ preserves an épinglage. Then d has

order $n < \infty$. Let $\tilde{G} = G^0 \times C_n$ (semidirect product, C_n a cyclic group of order $n \geq 1$ with generator e) with group structure $(a, e^t)(a', e^{t'}) = (a\mathrm{Ad}(d)^t(a'), e^{t+t'})$. Now $\pi : \tilde{G} \to G, \pi(g, e^t) = gd^t$ is a group homomorphism with kernel $K = \{(d^t, e^{-t}); d^t \in G^0\}$; it induces an isomorphism $\tilde{G}^0 \xrightarrow{\sim} G^0$. Let $x \in \tilde{G}$ be s.t. $\pi(x) = su$, x of the form $(x_0, e), x_0 \in G^0$. We have $(g, 1)x_s(g^{-1}, 1) = x_s(z, 1)k$, with $k \in K$. Taking images in $\tilde{G}/G^0$ we see that k goes to the neutral element hence $k \in G^0$. But $K \cap G^0 = \{1\}$ so that $k = 1$. We have $(g, 1)x_u(g^{-1}, 1) = x_u(y, 1)k'$ with $k' \in K$. As above we see that $k' = 1$. We have $(y, 1)x_s(y^{-1}, 1) = x_s k''$ with $k'' \in K$. As above we see that $k'' = 1$. From the definitions we see that the $\tilde{G}^0$-conjugacy class of x is cuspidal. Applying 17.7 to $\tilde{G}, x_s, x_u, (g, 1), (y, 1), (z, 1)$ instead of G, s, u, g, y, z we find $y' \in G^0$ such that $(y', 1)x_s(y'^{-1}, 1) = x_s$, $(y, 1) = x_u^{-1}(y', 1)x_u(y'^{-1}, 1)$. Applying π we get $y'sy'^{-1} = s, y = u^{-1}y'uy'^{-1}$. Thus Lemma 17.5 holds in the present case.

17.9. In the setup of 17.5 assume that G has no closed connected normal subgroup other than G^0 and $\{1\}$. Assume also that $p = 0$. We have $G^0 = \prod_{j \in \mathbf{Z}/b} H_j$ where H_j is connected, simply connected, almost simple, $b \geq 1$ and $sH_js^{-1} = H_{j+1}$, $u \in G^0$. Set $g = (g_j), z = (z_j), y = (y_j), u = (u_j)$ where $g_j \in H_j, z_j \in \mathcal{Z}_{H_j}$, $y_j \in H_j, u_j \in H_j$. We have $g_{j+1}^s = z_j g_j$, $g_j^{u_j} = y_j g_j$, $y_{j+1}^s = y_j$, $z_j^u = z_j$, $u_{j+1}^s = u_j$. Let G' be the subgroup of G generated by H_0, s^b. This is a closed subgroup with identity component H_0 since s has finite order (see 17.2; recall that su is isolated in G). We have $g_0^{u_0} = y_0 g_0, g_0^{s^b} = \mathbf{z} g_0$, where $\mathbf{z} = z_{b-1}^{s^{b-1}} \dots z_1^s z_0 \in \mathcal{Z}_{H_0}$. We have $y_0^{s^b} = y_0$. Also $s^b u_0 = u_0 s^b$. We show that the H_0-conjugacy class of $s^b u_0$ is cuspidal. By 17.3(a), it is enough to show that:

(i) the $Z_{H_0}(s^b)^0$-conjugacy class of u_0 is cuspidal in $Z_{G'}(s^b)^0$,

(ii) $s^b u_0$ is isolated in G'.

Since the G^0-conjugacy class of su is cuspidal, we see from 17.3(a) that:

(iii) the $Z_G(s)^0$-conjugacy class of u is cuspidal in $Z_G(s)$,

(iv) su is isolated in G.

Now $Z_G(s)^0 = Z_{G^0}(s)$ (see 17.4(b)) consists of all (x_j) where $x_j \in H_j$ satisfy $x_{j+1}^s = x_j$. We may identify $Z_{G^0}(s) = Z_{H_0}(s^b)$ and (i) follows. We prove (ii). From (iv) we see that $\mathcal{Z}_{Z_{G^0}(s)} \cap Z_{G^0}(u)$ is finite. Hence $f \in Z_{H_0}(s^b)$ subject to $f^{u_0} = f$ has finitely many possible values. Hence (iv) holds.

Applying 17.8 to $G', s^b, u_0, g_0, y_0, \mathbf{z}$ instead of G, s, u, g, y, z we find $\tilde{y} \in H_0$ such that $\tilde{y}^{s^b} = \tilde{y}$, $y_0 = \tilde{y}^{u_0}\tilde{y}^{-1}$. Set $y'_j = \tilde{y}^{s^{-j}} \in H_j$. Clearly, ${y'_{j+1}}^s = y'_j, y_0 y'_0 = {y'_0}^{u_0}$. Hence $y_j y'_j = {y'_j}^{u_j}$. (We have

$y_j y'_j = (y_0 y'_0)^{s^{-j}} = ({y'_0}^{u_0})^{s^{-j}} = ({y'_0}^{s^{-j}})^{u_j} = {y'_j}^{u_j}$.)

Hence setting $y' = (y'_j)$ we have $y'^s = y'$, $y = y'^u y'^{-1}$. Thus Lemma 17.5 holds in the present case.

17.10. In the setup of 17.5 assume that

(a) *G has no closed connected normal subgroup other than $G^0, \{1\}$ and $p > 1$.*

We have $G^0 = \prod_{i \in \mathbf{Z}/a, j \in \mathbf{Z}/b} H_{ij}$ where H_{ij} is connected, simply connected, almost

simple, $a \geq 1, b \geq 1$ and $uH_{ij}u^{-1} = H_{i+1,j}$, $sH_{ij}s^{-1} = H_{i,j+1}$. Let G' be the subgroup of G generated by H_{00}, u^a, s^b. This is a closed subgroup with identity component H_{00} since s has finite order (see 17.2; recall that su is isolated) and u has finite order, power of p. Now a is a power of p and b is prime to p. Set $g = (g_{ij}), z = (z_{ij}), y = (y_{ij})$ where $g_{ij} \in H_{ij}$, $z_{ij} \in \mathcal{Z}_{H_{ij}}$, $y_{ij} \in H_{ij}$. We have

$g^s_{i,j+1} = z_{ij}g_{ij}$, $g^u_{i+1,j} = y_{ij}g_{ij}$, $y^s_{i,j+1} = y_{ij}$, $z^u_{i+1,j} = z_{ij}$.

(The last equation follows from $uz = zu$: we have

$z^u g^u = (g^s)^u = (g^u)^s = (yg)^s = yzg = zyg = zg^u$

hence $z^u = z$.) We have

$g_{00}^{u^a} = \mathbf{y}g_{00}, \mathbf{y} = y_{a-1,0}^{u^{a-1}} \dots y^u_{1,0}y_{0,0} \in H_{00}$,

$g_{00}^{s^b} = \mathbf{z}g_{00}, \mathbf{z} = z_{0,b-1}^{s^{b-1}} \dots z^s_{0,1}z_{0,0} \in \mathcal{Z}_{H_{00}}$.

Since $y_{ij}^{s^b} = y_{ij}$ we have $\mathbf{y}^{s^b} = \mathbf{y}$. Also, $s^bu^a = u^as^b$. We show that the H_{00}-conjugacy class of s^bu^a is cuspidal. By 17.3(a) it is enough to show that:

(i) the $Z_{H_{00}}(s^b)^0$-conjugacy class of u^a is cuspidal in $Z_{G'}(s^b)^0$,

(ii) s^bu^a is isolated in G'.

Since the G^0-conjugacy class of su is cuspidal, we see from 17.3(a) that:

(iii) the $Z_G(s)^0$-conjugacy class of u is cuspidal in $Z_G(s)$,

(iv) su is isolated in G.

Now $Z_G(s)^0 = Z_{G^0}(s)$ (see 17.4(b)) consists of all (x_{ij}) where $x_{ij} \in H_{ij}$ satisfy $x^s_{i,j+1} = x_{ij}$. For each $i \in \mathbf{Z}/a$ let $F_i = \prod_{j \in \mathbf{Z}/b} H_{ij}$. Then

$sF_is^{-1} = F_i$, $uF_iu^{-1} = F_{i+1}$, $Z_{G^0}(s) = \prod_{i \in \mathbf{Z}/a} Z_{F_i}(s)$.

By an argument in 12.5(e) applied to $Z_G(s), Z_G(s)^0 = \prod_i Z_{F_i}(s), u$ instead of $G, G^0 = \prod_i H_i, u$, we see that the $Z_{F_0}(s)$-conjugacy class of u^a is cuspidal in the subgroup generated by $Z_{F_0}(s), u^a$. We may identify $Z_{F_0}(s) = Z_{H_{00}}(s^b)$ and (i) follows. We prove (ii). From (iv) we see that $\mathcal{Z}_{Z_{G^0}(s)} \cap Z_{G^0}(u)$ is finite. Hence if (f_i) satisfies $f_i \in \mathcal{Z}_{Z_{F_i}(s)}, f^u_{i+1} = f_i$ then f_i has finitely many values. Hence $f_0 \in \mathcal{Z}_{Z_{F_0}(s)}$, subject to $f_0^{u^a} = f_0$ has finitely many values. Hence $f_0 \in Z_{H_{00}}(s^b)$ subject to $f_0^{u^a} = f_0$ has finitely many values. Hence (ii) holds.

Applying 17.8 to $G', s^b, u^a, g_{00}, \mathbf{y}, \mathbf{z}$ instead of G, s, u, g, y, z we find $\tilde{y} \in H_{00}$ such that $\tilde{y}^{s^b} = \tilde{y}$, $\mathbf{y} = \tilde{y}^{u^a}\tilde{y}^{-1}$. Set $y'_{ij} \in H_{ij}$ by $y'_{0j} = \tilde{y}^{s^{-j}}$ for $j \in \mathbf{Z}/b$,

$y'_{ij} = y_{i-1,0}^{u^{-1}s^{-j}} \dots y_{1,0}^{u^{-i+1}s^{-j}} y_{0,0}^{u^{-i}s^{-j}} \tilde{y}^{u^{-i}s^{-j}}$

for $i = 1, \dots, a-1$ and $j \in \mathbf{Z}/b$. Clearly, ${y'_{i,j+1}}^s = y'_{ij}$. Moreover, $y_{ij}y'_{ij} = {y'_{i+1,j}}^u$ for $i = 0, 1, \dots, a-2$. The same holds for $i = a-1$:

$$\begin{aligned} y_{a-1,j}y'_{a-1,j} &= y_{a-1,0}^{s^{-j}} y_{a-2,0}^{u^{-1}s^{-j}} \dots y_{1,0}^{u^{-a+2}s^{-j}} y_{0,0}^{u^{-a+1}s^{-j}} \tilde{y}^{u^{-a+1}s^{-j}} \\ &= \mathbf{y}^{u^{-a+1}s^{-j}} \tilde{y}^{u^{-a+1}s^{-j}} = \tilde{y}^{us^{-j}} = {y'_{0j}}^u = {y'_{aj}}^u. \end{aligned}$$

Hence setting $y' = (y'_{ij}) \in G^0$ we have $y'^s = y', y = y'^u y'^{-1}$. Thus Lemma 17.5 holds in the present case.

17.11. We now prove Lemma 17.5 by induction on $\dim G$. If $\dim G = 0$ the result is trivial. We now assume that $\dim G > 0$. Assume first that $G^0 = G_1 \times G_2$ where

$G_i \ne \{1\}$ are connected, simply connected, normal in G. Let $G'_1 = G/G_2, G'_2 = G/G_1, G' = G'_1 \times G'_2$. Then $G \subset G', G^0 = G'^0$. We have

$s = (s_1, s_2), u = (u_1, u_2), z = (z_1, z_2), g = (g_1, g_2), y = (y_1, y_2)$

where s_i is semisimple in G'_i, u_i is unipotent in G'_i, $z_i \in \mathcal{Z}_{G_i}$, $g_i \in G_i$, $y_i \in Z_{G'_i}(s_i)^0$. We have $s_i u_i = u_i s_i$, $g_i^{s_i} = z_i g_i$, $g_i^{u_i} = y_i g_i$. Also the G_i-conjugacy class of $s_i u_i$ is cuspidal. By the induction hypothesis, we can find $y'_i \in Z_{G'_i}(s_i)^0$ with $y_i = y_i'^{u_i} y_i'^{-1}$. Let $y' = (y'_1, y'_2)$. Then $y' \in Z_{G^0}(s)$, $y = y'^u y'^{-1}$. Thus Lemma 17.5 holds in the present case.

Next we assume that no decomposition $G^0 = G_1 \times G_2$ as above exists. Then the result follows from 17.9, 17.10. The lemma is proved.

Lemma 17.12. *Let $s \in G$ be semisimple and $u \in G$ be unipotent such that $su = us$ and the G^0-conjugacy class of su is cuspidal. Assume that $g \in G^0$, $g^s = g, g^u = yg$ with $y \in Z_G(s)^0$. Then $y = y'^u y'^{-1}$ for some $y' \in Z_G(s)^0$.*

Assume first that G^0 is semisimple and that

(a) *there exists an element d in the connected component of G that contains su such that $\mathrm{Ad}(d) : G^0 \to G^0$ preserves an épinglage and such that $\{d^t; t \in \mathbf{Z}\} \cap G^0 = \{1\}$.*

Using [St, 9.16] we can find a reductive group $\tilde G$ with $\tilde G$ semisimple, simply connected and a surjective homomorphism of algebraic groups $\pi : \tilde G \to G$ such that $K = \mathrm{Ker}\pi \subset \mathcal{Z}_{\tilde G^0}$. Pick $\tilde s' \in \tilde G$ semisimple, $\tilde u \in \tilde G$ unipotent such that $\pi(\tilde s') = s, \pi(\tilde u) = u$. Then $\tilde u \tilde s' = \tilde s' \tilde u k$ with $k \in K$. Now $\tilde u k = \tilde s'^{-1} \tilde u \tilde s'$ is unipotent. Since $\tilde u$ normalizes K (a diagonalizable group) it follows that $\tilde u k = k' \tilde u k'^{-1}$ for some $k' \in K$. Set $\tilde s = \tilde s' k'$. Then $\tilde s$ is semisimple (since $\tilde s'$ normalizes K, a diagonalizable group) and $\pi(\tilde s) = s$, $\tilde u \tilde s = \tilde s \tilde u$. Since $\pi(\tilde s \tilde u) = su$ we see that the G^0-conjugacy class of su is cuspidal. Let $g' \in \tilde G^0$ be such that $\pi(g') = g$. Since $\hat Z = \{x \in \tilde G; \tilde s x \in x \tilde s \mathcal{Z}_{\tilde G^0}\}$ has identity component $Z_{\tilde G}(\tilde s)^0$, we see that $Z_{\tilde G}(\tilde s)^0 \to Z_G(s)^0$ is surjective. Hence we can find $\tilde y \in Z_{\tilde G}(\tilde s)^0$ such that $\pi(\tilde y) = y$. We have $\pi(g'^{\tilde u}) = \pi(\tilde y g')$ hence $g'^{\tilde u} = k' \tilde y g'$ for some $k' \in \Gamma$. Hence $g' \tilde u g'^{-1} = \tilde u k' \tilde y$, equality in $\hat Z$. Hence $\tilde u k'$ is unipotent in $\hat Z / \hat Z^0$. Since the image of $\tilde u k'$ in $\hat Z / \hat Z^0$ normalizes the image of K in $\hat Z / \hat Z^0$, we see that there exists $k_1 \in K$ such that $\tilde u k' = k_1 \tilde u k_1^{-1}$ (equality in $\hat Z / \hat Z^0$) hence $\tilde u k' = k_1 \tilde u k_1^{-1} \tilde y_1$ (equality in $\hat Z$) for some $\tilde y_1 \in Z_{\tilde G}(\tilde s)^0$. Hence $g' \tilde u g'^{-1} = k_1 \tilde u k_1^{-1} \tilde y_1 \tilde y = k_1 \tilde u \tilde y'' k_1^{-1}$ for some $\tilde y'' \in Z_{\tilde G}(\tilde s)^0$. Set $\tilde g = k_1^{-1} g'$. Then $\tilde g \in \tilde G^0$, $\tilde g^{\tilde u} = \tilde y'' \tilde g$, $\pi(\tilde g) = g$. We have $\pi(\tilde g^{\tilde s}) = \pi(\tilde g)$ hence $\tilde g^{\tilde s} = z \tilde g$ for some $z \in K$. Applying Lemma 17.5 to $\tilde G, \tilde s, \tilde u, \tilde g, \tilde y'', z$ instead of G, s, u, g, y, z we see that $\tilde y'' = \tilde y'^{\tilde u} \tilde y'^{-1}$ for some $\tilde y' \in Z_{\tilde G}(\tilde s)^0$. Let $y' = \pi(\tilde y')$. Then $\pi(\tilde y'') = y'^u y'^{-1}$, $y' \in Z_G(s)^0$. Also, $g^u = \pi(\tilde y'') g, g^u = yg$ hence $\pi(\tilde y'') = y$ and $y = y'^u y'^{-1}$. Thus the lemma holds in the present case.

Next assume that G^0 is semisimple and G is generated by a connected component D. We can find $d \in D$ such that $\mathrm{Ad}(d) : G^0 \to G^0$ preserves an épinglage. Then d has order $n < \infty$. Let $G' = G^0 \times C_n$ (semidirect product, C_n a cyclic group of order $n \ge 1$ with generator e) with group structure

$(a, e^t)(a', e^{t'}) = (a\mathrm{Ad}(d)^t(a'), e^{t+t'})$. Now $\pi' : G' \to G, \pi(g, e^t) = gd^t$ is a group homomorphism with kernel $K' = \{(d^t, e^{-t}); d^t \in \mathcal{Z}_{G^0}\}$; it induces an isomorphism $G'^0 \xrightarrow{\sim} G^0$. Then G' is as in the first part of the proof. Let $x \in G'$ be such that $\pi'(x) = su$. From the definitions we see that the G'^0-conjugacy class of x is cuspidal. For any $h \in Z_G(s)$ we have $(h, 1)x_s(h^{-1}, 1) = x_s k$ with $k \in K'$. Taking images in G'/G'^0 we see that k goes to 1 hence $k \in G'^0 \cap K' = \{1\}$ hence $k = 1$. Thus, $(h, 1) \in Z_{G'}(x_s)$. It follows that for any $h \in Z_G(s)^0$ we have $(h, 1) \in Z_{G'}(x_s)^0$. In particular $(y, 1) \in Z_{G'}(x_s)^0$. In the same way we see that $(g, 1)x_s(g^{-1}, 1) = x_s$, $(g, 1)x_u(g^{-1}, 1) = x_u(y, 1)$ (compare 17.8). From the first part of the proof we see that there exists $(y', 1) \in Z_{G'}(x_s)^0$ such that $(y, 1) = x_u^{-1}(y', 1)x_u(y'^{-1}, 1)$. It follows that $y' \in Z_G(s)^0$, $y = u^{-1}y'uy'^{-1}$. Thus the lemma holds in the present case.

Next, assume that G^0 is semisimple (but there is no assumption on G/G^0). Let G_1 be the subgroup of G generated by the connected component that contains su. By the earlier part of the proof, the lemma holds for G_1, s, u, g, y instead of G, s, u, g, y. But then it automatically holds for G, s, u, g, y.

Finally, we consider the general case. Let $\pi'' : G \to \bar{G} = G/\mathcal{Z}_{G^0}^0$ be the obvious homomorphism. Let $\bar{s} = \pi''(s), \bar{u} = \pi''(u)$. Then the $\bar{G}^0$-conjugacy class of $\bar{s}\bar{u} = \bar{u}\bar{s}$ is cuspidal. Let $\bar{g} = \pi''(g) \in \bar{G}^0, \bar{y} = \pi''(y)$. Then $\bar{g}^{\bar{s}} = \bar{g}, \bar{g}^{\bar{u}} = \bar{y}\bar{g}$, $\bar{y} \in Z_{\bar{G}}(\bar{s})^0$. Since the lemma holds for $\bar{G}$ instead of G, we have $\bar{y} = \bar{y}'^{\bar{u}}\bar{y}'^{-1}$ for some $\bar{y}' \in Z_{\bar{G}}(\bar{s})^0$. Let $\tilde{Z} = \{x \in G; xsx^{-1} \in s\mathcal{Z}_{G^0}^0\}$. Then π'' induces a surjective homomorphism $\tilde{Z} \to Z_{\bar{G}}(\bar{s})$. Moreover, $\tilde{Z}^0 = Z_G(s)^0$ hence π'' induces a surjective homomorphism $Z_G(s)^0 \to Z_{\bar{G}}(\bar{s})^0$. Hence we can find $y_1' \in Z_G(s)^0$ such that $\pi''(y_1') = \bar{y}'$. We have $y = y_1'^{u}y_1'^{-1}z$ for some $z \in \mathcal{Z}_{G^0}^0$. Thus $uy = y_1'uy_1'^{-1}z$. Since $uy = gug^{-1}$ is unipotent, we see that $y_1'uy_1'^{-1}z$ is unipotent. Also, $z = y_1'(u^{-1}y_1'^{-1}u)y \in Z_G(s)^0$ (since $y, y_1' \in Z_G(s)^0, u \in Z_G(s)$) hence $z \in Z_G(s)^0 \cap \mathcal{Z}_{G^0}^0$.

Assume first that $p = 0$. We set $y' = y_1'$. Then $y'uy'^{-1}$ being unipotent is in G^0 hence it commutes with z. Since $y'uy'^{-1}z$ is unipotent, we have $z = 1$ and $y = y'^{u}y'^{-1}$, as required.

Next assume that $p > 1$. Then $y_1'uy_1'^{-1}$ has finite order and, being in $Z_G(s)$, it normalizes $H = Z_G(s)^0 \cap \mathcal{Z}_{G^0}^0$. Hence, if H' is the subgroup of G generated by H and $y_1'uy_1'^{-1}$, we see that H' contains H as a normal subgroup of finite index, a power of p. Since H is diagonalizable, it follows that any two unipotent elements of H' in the same H-coset are H-conjugate. In particular, the unipotent elements $uy = y_1'uy_1'^{-1}z, y_1'uy_1'^{-1}$ of H' are H-conjugate. Hence $uy = \zeta y_1'uy_1'^{-1}\zeta^{-1}$ for some $\zeta \in H$. We set $y' = \zeta y_1'$. Then $y' \in Z_G(s)^0$ and $uy = y'uy'^{-1}$. The lemma is proved.

Proposition 17.13. *Let* $\mathbf{c}$ *be a cuspidal* G^0*-conjugacy class in* G*. Let* $s \in G$ *be the semisimple part of some element of* $\mathbf{c}$ *and let* δ *be a connected component of* $Z_G(s)$*. Then* $\{u \in Z_G(s) \text{ unipotent }, u \in \delta, su \in \mathbf{c}\}$ *is a single* $Z_G(s)^0$*-conjugacy class.*

We must show that, if $u, u' \in Z_G(s)$ are unipotent, $u' \in uZ_G(s)^0$ and $su \in \mathbf{c}, su' \in \mathbf{c}$, then u, u' are $Z_G(s)^0$-conjugate. We can find $g \in G^0$ such that $gsug^{-1} = su'$. Then $gsg^{-1} = s, gug^{-1} = u' = uy$ where $y \in Z_G(s)^0$. Hence $g^s = g, g^u = yg$. By Lemma 17.12 we can find $y' \in Z_G(s)^0$ such that $y = u^{-1}y'uy'^{-1}$. Then $u' = y'uy'^{-1}$. The proposition is proved.

18. A property of isolated elements

18.1. This section contains a characterization of isolated elements of G. (This will not be used later in this Part.)

Proposition 18.2. *Let $s \in G$ be semisimple and $u \in G$ be unipotent such that $su = us$. Then su is isolated in G if and only if s is isolated in G. Equivalently, we have $(\mathcal{Z}_{Z_G(s)^0} \cap Z_G(u))^0 = (\mathcal{Z}_{G^0} \cap Z_G(su))^0$ if and only if $\mathcal{Z}^0_{Z_G(s)^0} = (\mathcal{Z}_{G^0} \cap Z_G(s))^0$.*

Clearly, if s is isolated in G then su is isolated in G. The proof of the converse is given in 18.3-18.12.

In 18.3-18.12, it is assumed that su is isolated in G.

18.3. In the setup of 18.2 assume that $u \in G^0$. (This condition is automatically satisfied if $p = 0$.) By assumption, we have $u \in Z_{G^0}(s)$. The image of u in $Z_{G^0}(s)/Z_G(s)^0$ is semisimple (by 17.4(a)) and unipotent hence is 1. Thus, $u \in Z_G(s)^0$. It follows that $(\mathcal{Z}_{Z_G(s)^0} \cap Z_G(u))^0 = \mathcal{Z}^0_{Z_G(s)^0}$ and $(\mathcal{Z}_{G^0} \cap Z_G(su))^0 = (\mathcal{Z}_{G^0} \cap Z_G(s))^0$. Hence the condition $(\mathcal{Z}_{Z_G(s)^0} \cap Z_G(u))^0 = (\mathcal{Z}_{G^0} \cap Z_G(su))^0$ implies that $\mathcal{Z}^0_{Z_G(s)^0} = (\mathcal{Z}_{G^0} \cap Z_G(s))^0$. Thus s is isolated in G.

18.4. In the setup of 18.2 assume that G^0 is semisimple, that $s \in G^0$. We show that s is isolated in G. We may assume that u is unipotent, quasi-semisimple in $Z_G(s)$. Let $\mathcal{Z} = \mathcal{Z}^0_{Z_G(s)^0}$. Assume that $\mathcal{Z} \neq \{1\}$. Let $L = Z_{G^0}\mathcal{Z}$, a Levi of a parabolic of G^0. Since $s \in G^0$, we have $\mathcal{Z}^0_L = \mathcal{Z}$. Since su is quasi-semisimple, we can find a Borel B and a maximal torus T of B that are normalized by su. Since $s \in G^0$, we have $s \in T$. Hence $T \subset Z_G(s)^0$ so that $\mathcal{Z} \subset T$ and $T \subset L$. Let $\beta = B \cap L$, a Borel of L. Let $\Pi \subset V = \mathrm{Hom}(T, \mathbf{k}^*) \otimes \mathbf{Q}$ be the set of simple roots of G^0 with respect to T, B (in particular, the corresponding root subgroups are contained in U_B). Let Q be the basis of $V^* = \mathrm{Hom}(\mathbf{k}^*, T) \otimes \mathbf{Q}$ dual to Π. There is a unique subset Q_1 of Q which is a basis for the subspace $V_1^* = \mathrm{Hom}(\mathbf{k}^*, \mathcal{Z}) \otimes \mathbf{Q}$ of V^*. Since $\mathcal{Z} \neq \{1\}$ we have $V_1^* \neq 0$ hence $Q_1 \neq \emptyset$. Now u normalizes $Z_G(s)^0$ hence $u\mathcal{Z}u^{-1} = \mathcal{Z}, uLu^{-1} = u, u\beta u^{-1} = \beta$. Hence the automorphism of V_1^* induced by $\mathrm{Ad}(u)$ preserves Q and Q_1 and the sum of elements in Q_1 is a non-zero $\mathrm{Ad}(u)$-invariant vector. Thus, $\mathrm{Ad}(u) : V_1^* \longrightarrow V_1^*$ has a non-zero fixed point set. It follows that $\dim(\mathcal{Z} \cap Z_G(u)) > 0$ contradicting the assumption that su is isolated in G. We have proved that $\mathcal{Z} = \{1\}$. Hence s is isolated in G.

18.5. In the setup of 18.2 assume that G^0 is semisimple, simply connected and 17.6(a) holds. Since m in 17.6(a) is 1 or a prime number, we have three cases:

(i) $u \in G^0, s \notin G^0$ and $1 < m \neq p$;

(ii) $u \notin G^0, s \in G^0$ and $1 < m = p$;
(iii) $u \in G^0, s \in G^0$ and $1 = m$.
In cases (ii),(iii) we have $s \in G^0$ hence by the argument in 18.4 we see that s is isolated in G. In cases (i),(iii) we have $u \in G^0$ hence by the argument in 18.3 we see that s is isolated in G.

18.6. In the setup of 18.2 assume that G^0 is semisimple, simply connected and 17.7(a) holds. Let $\pi : G \to \bar{G}$ be as in 17.7. Let $\bar{s} = \pi(s), \bar{u} = \pi(u)$. One checks that $\bar{s}\bar{u}$ is isolated in $\bar{G}$. Applying 18.5 to $\bar{G}, \bar{s}, \bar{u}$ instead of G, s, u we see that $\mathcal{Z}^0_{Z_{\bar{G}}(\bar{s})^0} = \{1\}$. Hence $\mathcal{Z}^0_{Z_G(s)^0} = \{1\}$. Thus s is isolated in G.

18.7. In the setup of 18.2 assume that G^0 is semisimple, simply connected and 17.8(a) holds. Let $\pi : \tilde{G} \to G, x \in \tilde{G}$ be as in 17.8. One checks that x is isolated in $\tilde{G}$. Applying 18.6 to $\tilde{G}, x_s, x_u$ instead of G, x, s we see that $\mathcal{Z}^0_{Z_{\tilde{G}}(\tilde{s})^0} = \{1\}$. Hence $\mathcal{Z}^0_{Z_G(s)^0} = \{1\}$. Thus s is isolated in G.

18.8. In the setup of 18.2 assume that G^0 is semisimple, simply connected and 17.10(a) holds. Let a, b, H_{ij}, G', F_k be as in 17.10. As in 17.10 we see that $s^b u^a$ is isolated in G'. Applying 18.7 to G', s^b, u^a instead of G, s, u we see that $\mathcal{Z}^0_{Z_{G'}(s^b)^0} = \{1\}$ that is $\mathcal{Z}^0_{Z_{H_{00}}(s^b)^0} = \{1\}$. Hence $\mathcal{Z}^0_{Z_{F_0}(s)^0} = \{1\}$. Now $u^k F_0 u^{-k} = F_k$ and since $us = su$, $\text{Ad}(u^k)$ is an isomorphism $Z_{F_0}(s)^0 \xrightarrow{\sim} Z_{F_k}(s)^0$ hence we have $\mathcal{Z}^0_{Z_{F_k}(s)^0} \cong \mathcal{Z}^0_{Z_{F_0}(s)^0} = \{1\}$. Hence $\mathcal{Z}^0_{Z_G(s)^0} = \prod_i \mathcal{Z}^0_{Z_{F_i}(s)^0} = \{1\}$. Thus s is isolated in G.

18.9. In the setup of 18.2 assume that G^0 is semisimple, simply connected. We show that s is isolated in G by induction on $\dim G$. If $\dim G = 0$ the result is trivial. We now assume that $\dim G > 0$. Assume first that $G^0 = G_1 \times G_2$ where $G_i \neq \{1\}$ are connected, simply connected, normal in G. Let G'_i, s_i, u_i be as in 17.11. Then $s_i u_i = u_i s_i$ is isolated in G'_i. By the induction hypothesis we have $\mathcal{Z}^0_{Z_{G_i}(s_i)^0} = \{1\}$ for $i = 1, 2$. Now $Z_G(s)^0 = Z_{G_1}(s_1)^0 \times Z_{G_2}(s_2)^0$ hence $\mathcal{Z}^0_{Z_G(s)^0} = \mathcal{Z}^0_{Z_{G_1}(s_1)^0} \times \mathcal{Z}^0_{Z_{G_2}(s_2)^0} = \{1\}$. Thus s is isolated in G.

Next we assume that no decomposition $G^0 = G_1 \times G_2$ as above exists. If $p > 1$, then 18.8 shows that s is isolated in G. If $p = 0$ then 18.3 shows that s is isolated in G. This completes the inductive proof.

18.10. In the setup of 18.2 assume that G^0 is semisimple and that 17.12(a) holds. Let $\pi : \tilde{G} \to G, \tilde{s}, \tilde{u}$ be as in 17.12. We show that $\tilde{s}\tilde{u}$ is isolated in $\tilde{G}$. Let $x \in (\mathcal{Z}^0_{Z_{\tilde{G}}(\tilde{s})^0} \cap Z_{\tilde{G}}(\tilde{u}))^0$. Then $\pi(x) \in \mathcal{Z}^0_{Z_G(s)^0} \cap Z_G(u)$. Indeed, the map $\pi : Z_{\tilde{G}}(\tilde{s})^0 \to Z_G(s)^0$ is a surjective, finite covering of connected reductive groups hence it restricts to a surjective map

(*) $\mathcal{Z}^0_{Z_{\tilde{G}}(\tilde{s})^0} \to \mathcal{Z}^0_{Z_G(s)^0}$.

We see that $\pi(x) \in (\mathcal{Z}^0_{Z_G(s)^0} \cap Z_G(u))^0$. Hence $\pi(x) \in (\mathcal{Z}^0_{G^0} \cap Z_G(u))^0$ and $\pi(x) = 1$. Hence $x \in \text{Ker}\pi$, a finite group. It follows that $x = 1$ and $\tilde{s}\tilde{u}$ is isolated in

$\tilde{G}$. Applying 18.9 to $\tilde{G}, \tilde{s}, \tilde{u}$ instead of G, s, u, we see that $\mathcal{Z}^0_{Z_{\tilde{G}}(\tilde{s})^0} = \{1\}$. Let $y \in (\mathcal{Z}_{Z_G(s)^0})^0$. By the surjectivity of $(*)$ we have $y = \pi(y')$ where $y' \in \mathcal{Z}^0_{Z_{\tilde{G}}(\tilde{s})^0}$. Hence $y' = 1$ and $y = 1$. We see that $(\mathcal{Z}_{Z_G(s)^0})^0 = \{1\}$. Thus s is isolated in G.

Next we assume, in the setup of 18.2 that G^0 is semisimple and G/G^0 is cyclic. Let $\pi' : G' \to G, x$ be as in 17.12. Then x is isolated in G'. By the argument above applied to G' instead of G we see that x_s is isolated in G'. It follows immediately that s is isolated in G.

18.11. In the setup of 18.2 assume that G^0 is semisimple. Let G_1 be the subgroup of G generated by the connected component that contains su. Clearly, su is isolated in G_1. Applying 18.10 to G_1, s, u instead of G, s, u we see that s is isolated in G_1. Hence s is isolated in G.

18.12. In the setup of 18.2 let $\pi' : G \to \bar{G}, \bar{s}, \bar{u}$ be as in 17.12. Then the $\bar{G}^0$-conjugacy class of $\bar{s}\bar{u} = \bar{u}\bar{s}$ is isolated in $\bar{G}$. Applying 18.11 to $\bar{G}, \bar{s}, \bar{u}$ instead of G, s, u we see that $\bar{s}$ is isolated in $\bar{G}$. Using now 2.3(a) we see that s is isolated in G. Proposition 18.2 is proved.

19. Properties of cuspidal local systems

19.1. Let $s \in G$ be semisimple and let $\mathfrak{c}$ be a unipotent $Z_G(s)^0$-conjugacy class in $Z_G(s)$. Assume that the unique G^0-conjugacy class $\mathbf{c}$ that contains $s\mathfrak{c}$ is isolated in G. Let $G_1 = \{g \in Z_{G^0}(s); g\mathbf{c}g^{-1} = \mathbf{c}\}$ (a subgroup of $Z_{G^0}(s)$ containing $Z_G(s)^0$). Let $\tilde{\mathbf{c}}$ be the variety of orbits for the $Z_G(s)^0$-action $z : (y, u) \mapsto (yz^{-1}, zuz^{-1})$ on $G^0 \times \mathfrak{c}$. Then $(y, u) \mapsto ysuy^{-1}$ is a finite principal covering $\pi : \tilde{\mathbf{c}} \to \mathbf{c}$ with group $G_1/Z_G(s)^0$. Let $\mathfrak{f}$ be a cuspidal local system on $\mathfrak{c}$. Let $\tilde{\mathfrak{f}}$ be the local system on $\tilde{\mathbf{c}}$ whose inverse image under $G^0 \times \mathfrak{c} \to \tilde{\mathbf{c}}$ is $\bar{\mathbf{Q}}_l \boxtimes \mathfrak{f}$. We show that

(a) $\pi_!\tilde{\mathfrak{f}}$ *is a cuspidal local system on* $\mathbf{c}$.

Since $\pi_!\tilde{\mathfrak{f}}$ is clearly a G^0-equivariant local system, it is enough to show that the local system $j^*\pi_!\tilde{\mathfrak{f}}$ on $\mathfrak{c}$ is cuspidal relative to $Z_G(s)$ (here $j : \mathfrak{c} \to \mathbf{c}$ is $u \mapsto su$); see 17.3(a). From the definitions we see that $j^*\pi_!\tilde{\mathfrak{f}} \cong \oplus_{g_1} \mathrm{Ad}(g_1)^*\mathfrak{f}$ where g_1 runs over a set of representatives for the $Z_G(s)^0$-cosets in G_1. Clearly, each $\mathrm{Ad}(g_1)^*\mathfrak{f}$ is a cuspidal local system on $\mathfrak{c}$ and (a) follows.

19.2. *Let $u \in G$ be unipotent, quasi-semisimple. Then $Z_{G^0}(u)$ is connected.*
(See [DM, 1.28]).

19.3. Let P be a parabolic of G^0 and let $x \in N_G P, v \in U_P, x' = xv$. We show that

(a) *there exists $v' \in U_P$ such that $x'_s = v'x_sv'^{-1}$, $x'_u = v'x_uv'^{-1}$* mod $Z_G(x'_s)^0$.

By 1.4(a) we can find Levi subgroups $L.L'$ of P such that $x_s \in N_G L$, $x'_s \in N_G L'$. Applying the canoical projection $p : N_G P \to N_G P/U_P$ to $x_s x_u v = x'_s x'_u$ we obtain $p(x_s)p(x_u) = p(x'_s)p(x'_u)$. Using the uniqueness of the Jordan decomposition in $N_G P/U_P$ we get $p(x_s) = p(x'_s)$. We can find $v' \in U_P$ such that $L' = v'Lv'^{-1}$. Then $v'x_sv'^{-1}, x'_s$ are elements of $N_G L' \cap N_G P$ with the same image under p hence, by

1.26(a), we have $v'x_sv'^{-1}=x'_s$. We must show that $x_u^{-1}v'^{-1}x'_uv'\in Z_G(x_s)^0$. We have $x'_u=v'x_s^{-1}v'^{-1}x_sx_uv$ hence $x_u^{-1}v'^{-1}x'_uv'=x_u^{-1}x_s^{-1}v'^{-1}x_sx_uvv'\in U_P$ since $x=x_sx_u\in N_G(U_P)$. Since $x_u^{-1}v'^{-1}x'_uv'\in Z_G(x_s)$ we see that $x_u^{-1}v'^{-1}x'_uv'\in Z_G(x_s)\cap U_P\subset Z_G(x_s)^0$ (we use 1.11).

19.4. Let C be an isolated stratum of G and let $\mathcal{E}\in\mathcal{S}(C)$. We show that conditions (i),(ii) below are equivalent.

(i) $\mathcal{E}$ is a cuspidal local system on C;

(ii) for any G^0-conjugacy class $\mathbf{c}$ in C, $\mathcal{E}|_{\mathbf{c}}$ is a cuspidal local system on $\mathbf{c}$.

Let P be a parabolic of G^0 with $P\ne G^0$ and let R be a U_P-coset in N_GP. By 19.3(a), the semisimple part of any element of R is contained in a fixed G^0-conjugacy class. Hence R is contained in a union of finitely many G^0-conjugacy classes. Hence $C\cap R$ is contained in a union of finitely many G^0-conjugacy classes in C; this union is necessarily disjoint (as a variety), by the definition of C. Thus, $C\cap R=\sqcup_{i=1}^n(\mathbf{c}_i\cap R)$ where $\mathbf{c}_i$ are G^0-conjugacy classes in C. Let d be the dimension of any G^0-conjugacy class in C minus the dimension of the P/U_P-conjugacy class of R/U_P in N_GP/U_P. If (ii) holds then $H_c^d(C\cap R,\mathcal{E})=\oplus_{i=1}^nH_c^d(\mathbf{c}_i\cap R,\mathcal{E}|_{\mathbf{c}_i})=0$ hence (i) holds. Conversely, assume that (i) holds and $\mathbf{c}$ is a G^0-conjugacy class in C. We must show that $H_c^d(\mathbf{c}\cap R,\mathcal{E}|_{\mathbf{c}})=0$ for R as above. We may assume that $\mathbf{c}\cap R\ne\emptyset$ hence $\mathbf{c}=\mathbf{c}_i$ for some i. We have $0=H_c^d(C\cap R,\mathcal{E})=\oplus_{i=1}^nH_c^d(\mathbf{c}_i\cap R,\mathcal{E}|_{\mathbf{c}_i})$ hence each $H_c^d(\mathbf{c}_i\cap R,\mathcal{E}|_{\mathbf{c}_i})$ is 0. In particular, $H_c^d(\mathbf{c}\cap R,\mathcal{E}|_{\mathbf{c}})=0$, as desired.

19.5. Let C be an isolated stratum of G. Let $\mathbf{c}$ be a G^0-conjugacy class in C. Let $\mathcal{F}$ be a G^0-equivariant cuspidal local system on $\mathbf{c}$. Let D be the connected component of G that contains C and let $T={}^D\mathcal{Z}^0_{G^0}$. Let $\mathcal{L}\in\mathcal{S}(T)$. Define $\pi:T\times\mathbf{c}\to C$ by $\pi(z,c)=zc$. We show that

(a) $\pi_!(\mathcal{L}\boxtimes\mathcal{F})\in\mathcal{S}(C)$ *is a cuspidal local system.*

Let Γ be the set of all $z\in T$ such that $z\mathbf{c}=\mathbf{c}$ (a finite group, see 1.23(a).) Then π is a finite principal covering with group Γ. Hence $\pi_!(\mathcal{L}\boxtimes\mathcal{F})$ is a local system on C. It is immediate that $\pi_!(\mathcal{L}\boxtimes\mathcal{F})\in\mathcal{S}(C)$. We show that it is cuspidal. Let P be a parabolic of G^0 with $P\ne G^0$ and let R be a U_P-coset in N_GP. Let d be $\dim\mathbf{c}$ minus the dimension of the P/U_P-conjugacy class of R/U_P in N_GP/U_P. We must show that $H_c^d(C\cap R,\pi_!(\mathcal{L}\boxtimes\mathcal{F}))=0$ or equivalently that $H_c^d(\pi^{-1}(C\cap R),\mathcal{L}\boxtimes\mathcal{F})=0$. Now $\pi^{-1}(C\cap R)=\{(z,c)\in T\times\mathbf{c};zc\in R\}$. By 19.3(a), the semisimple part of any element of R is contained in a fixed G^0-conjugacy class. Hence for $(z,c)\in\pi^{-1}(C\cap R)$, $(zc)_s$ is contained in a fixed semisimple G^0-conjugacy class hence zc is contained in a union of finitely many G^0-conjugacy classes, hence z can take only finitely many values. Thus there exist $z_1,z_2,\dots,z_m$ in $T\times\mathbf{c}$ such that

$$\pi^{-1}(C\cap R)=\sqcup_{i=1}^m\{(z_i,c);c\in\mathbf{c}\cap z_i^{-1}R\},$$

$$H_c^d(\pi^{-1}(C\cap R),\mathcal{L}\boxtimes\mathcal{F})\cong\oplus_{i=1}^m\mathcal{L}_{z_i}\otimes H_c^d(\mathbf{c}\cap z_i^{-1}R,\mathcal{F})=0.$$

This proves (a).

19.6. Let H be a connected algebraic group acting transitively on the variety X. Assume that we are given $\mathbf{F}_q$-rational structures on H, X compatible with the action. Let $F : H \longrightarrow H$, $F : X \longrightarrow X$ be the Frobenius maps. Let Υ be a set of representatives for the isomorphism classes of irreducible H-equivariant local systems $\mathcal{F}$ on X such that $F^*\mathcal{F} \cong \mathcal{F}$. For any $\mathcal{F} \in \Upsilon$ we choose $\phi : F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. Then $\chi_{\mathcal{F},\phi} : X^F \longrightarrow \bar{\mathbf{Q}}_l$ is a function constant on the orbits of H^F, independent of the choice of ϕ, up to a non-zero scalar.

Lemma 19.7. *$(\chi_{\mathcal{F},\phi})_{\mathcal{F}\in\Upsilon}$ is a $\bar{\mathbf{Q}}_l$-basis of the vector space of functions $X^F \longrightarrow \bar{\mathbf{Q}}_l$ that are constant on the orbits of H^F.*

A special case of this (when H is reductive and X is a unipotent conjugacy class in H) is proved in [L13, §24, p.140]. A similar proof works in the general case. We can find $x \in X^F$. Let $H_x = \{h \in H; hx = x\}$. Associating to $\mathcal{F} \in \Upsilon$ the stalk $\mathcal{F}_x$ (an irreducible H_x-module, by the equivariance of $\mathcal{F}$, on which H_x acts through its finite quotient $\Gamma = H_x/H_x^0$) gives a bijection between Υ and a set Υ' of representatives for the isomorphism classes of irreducible $\bar{\mathbf{Q}}_l[\Gamma]$-modules V such that there exists an isomorphism $\iota_V : V \longrightarrow V$ with $\iota_V\gamma = F(\gamma)\iota_V : V \longrightarrow V$ for all $\gamma \in \Gamma$. (For $V = \mathcal{F}_x$ we may take ι_V to be the isomorphism $\mathcal{F}_x \longrightarrow \mathcal{F}_x$ induced by ϕ^{-1}.) Now F acts naturally on Γ and, according to [SS, 2.7],

(a) $H_x \longrightarrow H^F\backslash X^F$, $z \mapsto H^F -$ *orbit of* hx *where* $h \in H, h^{-1}F(h) = z$ *induces a bijection between the set of F-twisted conjugacy classes in Γ and the set $H^F\backslash X^F$ of H^F-orbits on X^F.*

Via this bijection, giving a function $X^F \longrightarrow \bar{\mathbf{Q}}_l$ that is constant on H^F-orbits is the same as giving a function $\Gamma \longrightarrow \bar{\mathbf{Q}}_l$ that is constant on F-twisted conjugacy classes in Γ. If $\mathcal{F} \in \Upsilon$ and $V = \mathcal{F}_x$, then the function $\chi_{\mathcal{F},\phi} : X^F \longrightarrow \bar{\mathbf{Q}}_l$ corresponds to the function

$$\gamma \mapsto \mathrm{tr}(\mathcal{F}_{hx} \xrightarrow{F(h)^{-1}} \mathcal{F}_x \xrightarrow{\phi} \mathcal{F}_x \xrightarrow{h} \mathcal{F}_{hx}) = \mathrm{tr}(\mathcal{F}_x \xrightarrow{F(h)^{-1}h} \mathcal{F}_x \xrightarrow{\phi} \mathcal{F}_x)$$
$$= \mathrm{tr}(\iota_V^{-1}\gamma^{-1}, V)$$

where $h \in H$ is such that $h^{-1}F(h) \in H_x$ has image γ in Γ. (We use the fact that, for any $h' \in H, y \in X$, the compositions

$$\mathcal{F}_{F(y)} \xrightarrow{\phi} \mathcal{F}_y \xrightarrow{h'} \mathcal{F}_{h'y}, \quad \mathcal{F}_{F(y)} \xrightarrow{F(h')} \mathcal{F}_{F(h')F(y)} \xrightarrow{\phi} \mathcal{F}_{h'y}$$

coincide.) It is then enough to show that the functions $\gamma \mapsto \mathrm{tr}(\iota_V^{-1}\gamma^{-1}, V)$ (for various $V \in \Upsilon'$) form a basis for the vector space of functions $\Gamma \longrightarrow \bar{\mathbf{Q}}_l$ that are constant on F-twisted conjugacy classes. This follows from a variant of the Schur orthogonality relations. (It also follows from 20.4(f) applied to the group algebra $\mathbf{E} = \bar{\mathbf{Q}}_l[\Gamma]$; in this case all elements of Γ are effective, see 20.4.)

19.8. In the remainder of this section we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$ and that G has a fixed $\mathbf{F}_q$-rational structure with Frobenius map $F : G \longrightarrow G$.

Let C be an isolated stratum of G such that $F(C) = C$. Let $\mathcal{I}$ be a set of representatives for the isomorphism classes of irreducible local systems $\mathcal{E}$ in $\mathcal{S}(C)$ such that $F^*\mathcal{E} \cong \mathcal{E}$; for each $\mathcal{E} \in \mathcal{I}$ we choose an isomorphism $\phi : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. We show:

(a) *the functions $\chi_{\mathcal{E},\phi}$ where $\mathcal{E} \in \mathcal{I}$ form a basis of the vector space of functions $C^F \to \bar{\mathbf{Q}}_l$ that are constant on the G^{0F}-conjugacy classes in C^F.*

Let D be the connected component of G that contains C. Since $\mathcal{I}$ is a finite set, we can find an integer $n \geq 1$, invertible in $\mathbf{k}$, such that

$(*)$ $\mathcal{E} \in \mathcal{I} \implies \mathcal{E} \in \mathcal{S}_n(C)$, (see 5.2),

$(**)$ $z \in {}^D\mathcal{Z}^0_{G^0}, F(z) = z \implies z^n = 1$.

Applying Lemma 19.7 to the transitive action 5.2(a) (with n as above) of $H = {}^D\mathcal{Z}^0_{G^0} \times G^0$ on C, we see that the functions $\chi_{\mathcal{E},\phi}$ where $\mathcal{E}$ runs over the elements of $\mathcal{I}$ that belong to $\mathcal{S}_n(C)$ (or equivalently, $\mathcal{E}$ runs over $\mathcal{I}$, see $(*)$) form a basis of the vector space of functions $C^F \to \bar{\mathbf{Q}}_l$ that are constant on the H^F-orbits in C^F. By $(**)$, the H^F-orbits on C^F are the same as the G^{0F}-conjugacy classes in C^F. This proves (a).

19.9. For any isolated stratum C (resp. isolated G^0-conjugacy class $\mathbf{c}$) in G such that $F(C) = C$ (resp. $F(\mathbf{c}) = \mathbf{c}$) let $\mathcal{C}_{G^0}(C)$ (resp. $\mathcal{C}_{G^0}(\mathbf{c})$) be the subspace of the vector space of functions $C^F \to \bar{\mathbf{Q}}_l$ (resp. $\mathbf{c}^F \to \bar{\mathbf{Q}}_l$) spanned by the functions $\chi_{\mathcal{F},\epsilon}$ where $\mathcal{F}$ runs through a set of representatives for the isomorphism classes of irreducible cuspidal local systems on C (resp. on $\mathbf{c}$) such that $F^*\mathcal{F} \cong \mathcal{F}$ and $\epsilon : F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ is a fixed isomorphism. Clearly, the subspace $\mathcal{C}_{G^0}(C)$ (resp. $\mathcal{C}_{G^0}(\mathbf{c})$) is independent of choices. From 19.8(a) (resp. Lemma 19.7) we see that the functions $\chi_{\mathcal{F},\epsilon}$ (as above) form a basis of $\mathcal{C}_{G^0}(C)$ (resp. $\mathcal{C}_{G^0}(\mathbf{c})$): they are part of a basis of the vector space of all functions $C^F \to \bar{\mathbf{Q}}_l$ (resp. $\mathbf{c}^F \to \bar{\mathbf{Q}}_l$) that are constant on G^{0F}-conjugacy classes. From the definitions we see that:

(a) *if $\mathcal{F}$ is a (not necessarily irreducible) cuspidal local system on C (resp. $\mathbf{c}$) and $\epsilon : F^*\mathcal{F} \to \mathcal{F}$ is an isomorphism then $\chi_{\mathcal{F},\epsilon}$ belongs to $\mathcal{C}_{G^0}(C)$ (resp. $\mathcal{C}_{G^0}(\mathbf{c})$).*

19.10. Let C be an isolated stratum of G such that $F(C) = C$. For any G^0-conjugacy class $\mathbf{c}$ in C such that $F(\mathbf{c}) = \mathbf{c}$ then $f \mapsto f|_{\mathbf{c}^F}$ defines a linear map

(a) $\mathcal{C}_{G^0}(C) \to \mathcal{C}_{G^0}(\mathbf{c})$,

(To see that this map is well defined, it is enough to show that, if $\mathcal{F}$ is a cuspidal local system on C and $\epsilon : F^*\mathcal{F} \to \mathcal{F}$ is an isomorphism then $\chi_{\mathcal{F},\epsilon}|_{\mathbf{c}^F} \in \mathcal{C}_{G^0}(\mathbf{c})$. This follows from 19.4.) We now take the direct sum of the maps (a) where $\mathbf{c}$ runs over the F-stable G^0-conjugacy classes in C. We show that

(b) *the resulting linear map $\mathcal{C}_{G^0}(C) \to \oplus_{\mathbf{c}}\mathcal{C}_{G^0}(\mathbf{c})$ is an isomorphism.*

It is obvious that this map is injective. To show that it is surjective it is enough to verify the following statement:

for any F-stable G^0-conjugacy class $\mathbf{c}$ in C, any cuspidal local system $\mathcal{F}$ on $\mathbf{c}$ and any isomorphism $F^*\mathcal{F} \to \mathcal{F}$, there exists $f \in \mathcal{C}_{G^0}(C)$ such that $f|_{\mathbf{c}^F} = \chi_{\mathcal{F},\epsilon}$ and $f|_{\mathbf{c}'^F} = 0$ for any F-stable G^0-conjugacy class $\mathbf{c}'$ in C with $\mathbf{c}' \neq \mathbf{c}$.

Let D be the connected component of G that contains C. Let $T = {}^D\mathcal{Z}^0_{G^0}$. Let $\mathcal{J}$

be a set of representatives for the isomorphism classes of local systems $\mathcal{L}$ of rank 1 in $\mathcal{S}(T)$ such that $F^*\mathcal{L} \cong \mathcal{L}$. For each $\mathcal{L} \in \mathcal{J}$ there is a unique isomorphism $\phi_{\mathcal{L}}$: $F^*\mathcal{L} \to \mathcal{L}$ which induces the identity map on the stalk of $\mathcal{L}$ at 1. Then $\theta_{\mathcal{L}} = \chi_{\mathcal{L},\phi_{\mathcal{L}}}$ is a character $T^F \to \bar{\mathbf{Q}}_l^*$ and $\mathcal{L} \mapsto \theta_{\mathcal{L}}$ is a bijection $\mathcal{J} \xrightarrow{\sim} \mathrm{Hom}(T^F, \bar{\mathbf{Q}}_l^*)$. Define $\pi : T \times \mathbf{c} \to C$ by $\pi(z, c) = zc$. For any $\mathcal{L} \in \mathcal{J}$, $\pi_!(\mathcal{L} \boxtimes \mathcal{F})$ is naturally isomorphic with its inverse image under F (using $\phi_{\mathcal{L}} \boxtimes \epsilon$); let $\chi_{\pi_!(\mathcal{L}\boxtimes\mathcal{F}),?} : C^F \to \bar{\mathbf{Q}}_l$ be the corresponding characteristic function. From 19.5(a) we see that $\chi_{\pi_!(\mathcal{L}\boxtimes\mathcal{F}),?} \in \mathcal{C}_{G^0}(C)$. From the definitions we have

$\chi_{\pi_!(\mathcal{L}\boxtimes\mathcal{F}),?}(x) = \sum_{z\in T^F, c\in\mathbf{c}^F; zc=x} \theta_{\mathcal{L}}(z)\chi_{\mathcal{F},\epsilon}(c)$

for $x \in C^F$. Let $f = \sum_{\mathcal{L}\in\mathcal{J}} \chi_{\pi_!(\mathcal{L}\boxtimes\mathcal{F}),?} \in \mathcal{C}_{G^0}(C)$. For $x \in C^F$ we have

$$f(x) = \sum_{z\in T^F, c\in\mathbf{c}^F; zc=x} \sum_{\mathcal{L}\in\mathcal{J}} \theta_{\mathcal{L}}(z)\chi_{\mathcal{F},\epsilon}(c) = \sum_{z\in T^F, c\in\mathbf{c}^F; zc=x} |T^F|\delta_{z,1}\chi_{\mathcal{F},\epsilon}(c).$$

Thus $f(x) = |T^F|\chi_{\mathcal{F},\epsilon}(x)$ if $x \in \mathbf{c}^F$ and $f(x) = 0$ if $x \in C^F - \mathbf{c}^F$. This completes the proof of (b).

19.11. *If E is the set of unipotent quasi-semisimple elements in some F-stable connected component of G that contains unipotent elements then E^F is a single G^{0F}-conjugacy class.*
This follows from the fact that E is a homogeneous G^0-space (see 1.9(a)) defined over $\mathbf{F}_q$ in which the isotropy group of any point is connected (see 19.2).

19.12. Let $\mathbf{c}$ be a cuspidal G^0-conjugacy class in G. Let $\mathbf{c}_\bullet = \sigma_G(x)$ for any $x \in \mathbf{c}$; this is the set of all quasi-semisimple elements $g \in G$ such that for some $x \in \mathbf{c}$ we have $x_s = g_s$, $x_u \in Z_G(g_s)^0 g_u$. Let Z be the set of all pairs (s, c) where $s \in G$ is semisimple and c is a connected component of $Z_G(s)$ such that there exists a unipotent element $u \in Z_G(s)$ with $su \in \mathbf{c}, u \in c$. We have a diagram

$$\mathbf{c} \xrightarrow{a} Z \xleftarrow{b} \mathbf{c}_\bullet$$

where $a(x) = (x_s, Z_G(x_s)^0 x_u), b(g) = (g_s, Z_G(g_s)^0 g_u)$. Now G^0 acts transitively on $\mathbf{c}, Z, \mathbf{c}_\bullet$ compatibly with a, b. For any $y \in G$ let

$$H_{G^0}(y) = \{h \in G^0; hy_sh^{-1} = y_s, hy_uh^{-1} \in Z_G(y_s)^0 y_u\},$$

a closed subgroup of $Z_{G^0}(y_s)$ containing $Z_G(y_s)^0$.

(a) *Let $x \in \mathbf{c}, g \in \mathbf{c}_\bullet$ be such that $a(x) = b(g)$. Let $H = H_{G^0}(x) = H_{G^0}(g)$ (the stabilizer of $a(x) = b(g)$ in G^0). The map $Z_{G^0}(x)/Z_{G^0}(x)^0 \to H/H^0$ induced by a is surjective and the map $Z_{G^0}(g)/Z_{G^0}(g)^0 \to H/H^0$ induced by b is an isomorphism.*
We have $a(x) = b(g) = (x_s, Z_G(x_s)^0 x_u) = (g_s, Z_G(g_s)^0 g_u)$. Let $h \in H$. Then x, hxh^{-1} are elements of $\mathbf{c}$ with the same semisimple part x_s and their unipotent parts are contained in the same connected component of $Z_G(x_s)$. By 17.13,

there exists $z \in Z_G(x_s)^0$ such that $hx_uh^{-1} = zx_uz^{-1}$. Thus, $h = zh_1$ where $h_1 \in Z_{G^0}(x)$. We see that $H = Z_G(x_s)^0 Z_{G^0}(x)$. This proves the first assertion of (a).

If $h \in H$ then g_u, hg_uh^{-1} are unipotent quasi-semisimple elements of $Z_G(g_s)$ contained in the same connected component of $Z_G(g_s)$ hence, by 17.13, there exists $z \in Z_G(g_s)^0$ such that $hg_uh^{-1} = zg_uz^{-1}$. Thus, $h = zh_1$ where $h_1 \in Z_{G^0}(g)$. We see that $H = Z_G(g_s)^0 Z_{G^0}(g)$. This shows that b induces a surjective map $Z_{G^0}(g)/Z_{G^0}(g)^0 \xrightarrow{\sim} H/H^0$ and that the group of components of H is the same as the group of components of $Z_{G^0}(g)/(Z_{G^0}(g) \cap Z_G(g_s)^0) = Z_{G^0}(g)/(Z_{Z_G(g_s)^0}(g_u)$ which (by the connectedness of $Z_{Z_G(g_s)^0}(g_u)$, see 19.2) is the same as the group of components of $Z_{G^0}(g)$. Thus, the surjective map $Z_{G^0}(g)/Z_{G^0}(g)^0 \xrightarrow{\sim} H/H^0$ must be an isomorphism. This proves (a).

19.13. In the setup of 19.12, let $\mathcal{L}$ be an irreducible G^0-equivariant local system on Z. Since $H/H^0 = Z_{G^0}(g)/Z_{G^0}(g)^0$ (see 19.12(a)) is commutative (see 17.4(a)), $\mathcal{L}$ has rank 1. Let $\tilde{\mathcal{L}} = a^*\mathcal{L}$, a local system of rank 1 on $\mathbf{c}$. We show:

(a) *if $\mathcal{F}$ is a cuspidal local system on $\mathbf{c}$ then $\mathcal{F} \otimes \tilde{\mathcal{L}}$ is a cuspidal local system on $\mathbf{c}$.*

Let P be a parabolic of G^0 with $P \neq G^0$ and let $x \in \mathbf{c} \cap N_G P$. Let d be $\dim \mathbf{c}$ minus the dimension of the P/U_P-conjugacy class of xU_P in $N_G P/U_P$. We must show that $H_c^d(\mathbf{c} \cap xU_P, \mathcal{F} \otimes \tilde{\mathcal{L}}) = 0$. By our assumption we have $H_c^d(\mathbf{c} \cap xU_P, \mathcal{F}) = 0$. Hence it is enough to show that $\tilde{\mathcal{L}}|_{\mathbf{c} \cap xU_P} \cong \bar{\mathbf{Q}}_l$. Since $\tilde{\mathcal{L}} = a^*\mathcal{L}$, it is enough to show that there exists a subvariety V of Z such that

(b) $a(\mathbf{c} \cap xU_P) \subset V$, $\mathcal{L}|_V \cong \bar{\mathbf{Q}}_l$.

Let V be the U_P-orbit of $a(x)$ in Z (for the restriction of the G^0-action to U_P). For this V the first assertion of (b) holds by 19.3(a). We now show that for this V, the second assertion of (b) holds. It is enough to note that $\mathcal{L}|_V$ is a U_P-equivariant local system of rank 1 on the homogeneous U_P-space V in which the isotropy group of $a(x)$ that is, $U_P \cap Z_G(x_s)$, is connected (we use that x_s normalizes U_P, see 1.11). Thus (b), hence also (a), are proved.

19.14. We now assume that $F(\mathbf{c}) = \mathbf{c}$. Then Z and $\mathbf{c}$ are defined over $\mathbf{F}_q$ and we denote again by F the corresponding Frobenius maps.

(a) *The map $a_0 : Z^F \to \mathbf{c}^F$ (restriction of $a : Z \to \mathbf{c}$) is surjective; the map $b_0 : Z^F \to \mathbf{c}_\bullet^F$ (restriction of $b : Z \to \mathbf{c}_\bullet$) induces a bijection on the sets of G^{0F}-orbits.*

This follows immediately from 19.12(a) and 19.7(a). We have a partition $\mathbf{c}_\bullet^F = \sqcup \gamma$ where γ runs over the G^{0F}-orbits on $\mathbf{c}_\bullet^F$. For any γ we set $Z_\gamma^F = b_0^{-1}(\gamma), \mathbf{c}_\gamma^F = a_0(Z_\gamma^F)$. From (a) we see that $Z^F = \sqcup_\gamma Z_\gamma^F$ is the partition of Z^F into G^{0F}-orbits and that $\mathbf{c}^F = \sqcup_\gamma \mathbf{c}_\gamma^F$ is a partition of $\mathbf{c}^F$ into non-empty G^{0F}-stable subsets.

Let $f \in \mathcal{C}_{G^0}(\mathbf{c})$ and let γ be a G^{0F}-orbit on $\mathbf{c}_\bullet^F$. Define $f_\gamma : \mathbf{c}^F \to \bar{\mathbf{Q}}_l$ by $f_\gamma(x) = 1$ if $\gamma \in \mathbf{c}_\gamma^F, f_\gamma(x) = 0$ if $\gamma \in \mathbf{c}^F - \mathbf{c}_\gamma^F$. We show that

(b) $$f_\gamma f \in \mathcal{C}_{G^0}(\mathbf{c}).$$

We may assume that $f = \chi_{\mathcal{F},\epsilon}$ where $\mathcal{F}$ is a cuspidal local system on $\mathbf{c}$ and $\epsilon : F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ is an isomorphism. Define $\tilde{f}_\gamma : Z^F \to \bar{\mathbf{Q}}_l$ by $\tilde{f}_\gamma(z) = 1$ if $z \in Z^F_\gamma$, $\tilde{f}_\gamma(z) = 0$ if $z \in Z^F - Z^F_\gamma$. By 19.7 applied to the homogeneous G^0-space Z, there exist irreducible G^0-equivariant local systems $\mathcal{L}^i$, ($i \in [1,m]$ on Z and isomorphisms $e^i : F^*\mathcal{L}^i \xrightarrow{\sim} \mathcal{L}^i$ such that $\tilde{f}_\gamma = \sum_{i\in[1,m]} c_i\chi_{\mathcal{L}^i,e^i}$ where $c_i \in \bar{\mathbf{Q}}_l$. Each $\mathcal{L}^i$ has rank 1. Composing with $a_0 : \mathbf{c}^F \to Z^F$ we obtain $\tilde{f}_\gamma \circ a_0 = \sum_{i\in[1,m]} c_i\chi_{\mathcal{L}^i,e^i} \circ a_0$, that is $f_\gamma = \sum_{i\in[1,m]} c_i\chi_{\tilde{\mathcal{L}}^i,\tilde{e}^i}$ where $\tilde{\mathcal{L}}^i = a^*\mathcal{L}^i$ and $\tilde{e}^i : F^*\tilde{\mathcal{L}}^i \xrightarrow{\sim} \tilde{\mathcal{L}}^i$ is induced by e^i. Hence

$$f_\gamma f = \sum_{i\in[1,m]} c_i\chi_{\tilde{\mathcal{L}}^i,\tilde{e}^i}\chi_{\mathcal{F},\epsilon} = \sum_{i\in[1,m]} c_i\chi_{\mathcal{F}\otimes\tilde{\mathcal{L}}^i,\epsilon\otimes\tilde{e}^i}.$$

Using 19.13(a), we see that this belongs to $\mathcal{C}_{G^0}(\mathbf{c})$. This proves (b).

For any G^{0F}-orbit γ on $\mathbf{c}^F_\bullet$ we set

$$\mathcal{C}_{G^0,\gamma}(\mathbf{c}) = \{f \in \mathcal{C}_{G^0}(\mathbf{c}); f = 0 \text{ on } \mathbf{c}^F - \mathbf{c}^F_\gamma\}.$$

From (b) we see that:

(c) $$\mathcal{C}_{G^0}(\mathbf{c}) = \oplus_\gamma \mathcal{C}_{G^0,\gamma}(\mathbf{c}).$$

19.15. We now fix $g \in \mathbf{c}^F_\bullet$. We set $s = g_s$. Let

$\mathfrak{c} = \{u \in Z_G(s)^0 g_u; u \text{ unipotent}, su \in \mathbf{c}\}$.

From the definitions we see that $\mathfrak{c} \neq \emptyset$ and from 17.13 we see that $\mathfrak{c}$ is a single (unipotent) $Z_G(s)^0$-conjugacy class in $Z_G(s)$. We have $F(\mathfrak{c}) = \mathfrak{c}$ and $\mathfrak{c}$ carries some non-zero cuspidal local system (see 17.3(a)). Define $\mathcal{C}_{Z_G(s)^0}(\mathfrak{c})$ in terms of $Z_G(s), \mathfrak{c}$ in the same way as $\mathcal{C}_{G^0}(\mathbf{c})$ was defined in terms of $G, \mathbf{c}$. For any $f \in \mathcal{C}_{G^0}(\mathbf{c})$ we define $\bar{f} : \mathfrak{c}^F \to \bar{\mathbf{Q}}_l$ by $\bar{f}(u) = f(su)$. We claim that

$\bar{f} \in \mathcal{C}_{Z_G(s)^0}(\mathfrak{c})$.

We may assume that $f = \chi_{\mathcal{F},\epsilon}$ where $\mathcal{F}$ is a cuspidal local system on $\mathbf{c}$ and $\epsilon : F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ is an isomorphism. Let $\mathcal{F}' = j^*\mathcal{F}$ where $j : \mathfrak{c} \to \mathbf{c}, u \mapsto su$ and let $\epsilon' : F^*\mathcal{F}' \xrightarrow{\sim} \mathcal{F}'$ be the isomorphism induced by ϵ. By 17.3(a), $\mathcal{F}'$ is a cuspidal local system on $\mathfrak{c}$ and by 19.9(a) applied to $\mathfrak{c}$ instead of $\mathbf{c}$ we see that $\chi_{\mathcal{F}',\epsilon'} \in \mathcal{C}_{Z_G(s)^0}(\mathfrak{c})$. Clearly, $\chi_{\mathcal{F}',\epsilon'} = \bar{f}$ and our claim is verified.

From 17.13 it follows that $\mathfrak{c}^F$ is stable under conjugation by $H_{G^0}(g)^F$. It is clear that for $f, \bar{f}$ as above, $\bar{f}$ is constant on any $H_{G^0}(g)^F$-conjugacy class in $\mathfrak{c}^F$. Thus we have a well defined linear map

(a) $$\mathcal{C}_{G^0}(\mathbf{c}) \to \mathcal{C}_{H_{G^0}(g)}(\mathfrak{c}), f \mapsto \bar{f}$$

where $\mathcal{C}_{H_{G^0}(g)}(\mathfrak{c})$ is the space of functions in $\mathcal{C}_{Z_G(s)^0}(\mathfrak{c})$ that are constant on any $H_{G^0}(g)^F$-conjugacy class in $\mathfrak{c}^F$.

We show that the map (a) is surjective. Now $\mathcal{C}_{H_{G^0}(g)}(\mathfrak{c})$ is spanned by functions $f' : \mathfrak{c}^F \longrightarrow \bar{\mathbf{Q}}_l$ of the form

$$f'(u) = |Z_G(s)^{0F}|^{-1} \sum_{y \in H_{G^0}(g)^F} \chi_{\mathfrak{f},\epsilon}(y^{-1}uy)$$

where $\mathfrak{f}$ is a cuspidal local system on $\mathfrak{c}$ and $\epsilon : F^*\mathfrak{f} \xrightarrow{\sim} \mathfrak{f}$ is an isomorphism. It is enough to show that any such f' is in the image of the map (a). Let $\pi : \tilde{\mathbf{c}} \longrightarrow \mathbf{c}, \tilde{\mathfrak{f}}$ be as in 19.1. Now ϵ induces an isomorphism $\tilde{\epsilon} : F^*\pi_!\tilde{\mathfrak{f}} \longrightarrow \pi_!\tilde{\mathfrak{f}}$. Let $f = \chi_{\pi_!\tilde{\mathfrak{f}},\tilde{\epsilon}} : \mathbf{c}^F \longrightarrow \bar{\mathbf{Q}}_l$. Since $\pi_!\tilde{\mathfrak{f}}$ is a cuspidal local system on $\mathbf{c}$ (see 19.1(a)) we see that $f \in \mathcal{C}_{G^0}(\mathbf{c})$. From the definitions we have

$$\bar{f}(u) = f(su) = |Z_G(s)^{0F}|^{-1} \sum_{y \in G^{0F}, u' \in \mathfrak{c}^F; ysu'y^{-1} = su} \chi_{\mathfrak{f},\epsilon}(u')$$

for $u \in \mathfrak{c}^F$. For each y, u' in the sum we have $y \in Z_{G^0}(s)^F$ and $yu'y^{-1} = u$ hence $y \in H_{G^0}(g)^F, u' = y^{-1}uy$; hence

$$\bar{f}(u) = |Z_G(s)^{0F}|^{-1} \sum_{y \in H_{G^0}(g)^F} \chi_{\mathfrak{f},\epsilon}(y^{-1}uy).$$

We see that $\bar{f} = f'$. Thus the surjectivity of the map (a) is established.

Next we note that, if γ' is a G^{0F}-orbit on $\mathbf{c}_\bullet^F$ that does not contain g, then the restriction of the map (a) to $\mathcal{C}_{G^0,\gamma'}(\mathbf{c})$ is 0. Using this and the direct sum decomposition 19.14(c) we deduce that, if γ is the G^{0F}-orbit on $\mathbf{c}_\bullet^F$ that contains g, then (a) restricts to a surjective linear map

(b) $$\mathcal{C}_{G^0,\gamma}(\mathbf{c}) \longrightarrow \mathcal{C}_{H_{G^0}(g)}(\mathfrak{c}).$$

We show that this map is injective. Let $f, f' \in \mathcal{C}_{G^0,\gamma}(\mathbf{c})$ be such that $f(su) = f'(su)$ for any $u \in \mathfrak{c}^F$. Since $\mathbf{c}_\gamma^F = \{ysuy^{-1}; y \in G^{0F}, u \in \mathfrak{c}^F\}$ and f, f' are constant on G^{0F}-conjugacy classes it follows that $f = f'$ on $\mathbf{c}_\gamma^F$. Since f, f' are 0 on $\mathbf{c}^F - \mathbf{c}_\gamma^F$ it follows that $f = f'$, as desired. We see that

(c) *the map (b) is an isomorphism.*

20. Twisted group algebras

20.1. Let Γ be a finite group. Let $\mathbf{E}$ be a finite dimensional $\bar{\mathbf{Q}}_l$-vector space with a direct sum decomposition $\mathbf{E} = \oplus_{w \in \Gamma} \mathbf{E}_w$ with $\dim \mathbf{E}_w = 1$ for all w. Assume that on $\mathbf{E}$ we are given an associative algebra structure with 1 such that $\mathbf{E}_w\mathbf{E}_y = \mathbf{E}_{wy}$ for any $w, y \in \Gamma$. Then $1 \in \mathbf{E}_1 - \{0\}$. We choose a basis $\{b_w; w \in \Gamma\}$ of $\mathbf{E}$ such that $b_w \in \mathbf{E}_w$ for all w. Each b_w is invertible. We have $b_wb_y = \lambda(w,y)b_{wy}, b_y^{-1}b_w^{-1} = \lambda(w,y)^{-1}b_{wy}^{-1}$ with $\lambda(w,y) \in \bar{\mathbf{Q}}_l^*$ for any $w, y \in \Gamma$. We show that

(a) *the algebra* $\mathbf{E}$ *is semisimple.*
Let M be a finite dimensional $\mathbf{E}$-module and let M' be an $\mathbf{E}$-submodule of M. We must show that there exists an $\mathbf{E}$-submodule of M complementary to M'. Let $\pi : M \to M'$ be a $\bar{\mathbf{Q}}_l$-linear map such that $\pi(m') = m'$ for all $m' \in M'$. Define a $\bar{\mathbf{Q}}_l$-linear map $\tilde{\pi} : M \to M'$ by $\tilde{\pi}(m) = |\Gamma|^{-1} \sum_{w\in\Gamma} b_w^{-1}\pi(b_w m)$. For $m' \in M'$ we have $\tilde{\pi}(m') = |\Gamma|^{-1} \sum_{w\in\Gamma} b_w^{-1} b_w m' = m'$. We show that $\tilde{\pi}$ is $\mathbf{E}$-linear. It is enough to show that $b_y^{-1}\tilde{\pi}(b_y m) = \tilde{\pi}(m)$ for $m \in M$, $y \in \Gamma$. We have

$$|\Gamma| b_y^{-1}\tilde{\pi}(b_y m) = b_y^{-1} \sum_w b_w^{-1}\pi(b_w b_y m) = \sum_w b_y^{-1} b_w^{-1}\pi(\lambda(w,y) b_{wy} m)$$
$$= \sum_w \lambda(w,y)^{-1} b_{wy}^{-1}\pi(\lambda(w,y) b_{wy} m) = \sum_w b_{wy}^{-1}\pi(b_{wy} m) = |\Gamma|\tilde{\pi}(m),$$

as desired. Now $\mathrm{Ker}\tilde{\pi}$ is an $\mathbf{E}$-submodule of M complementary to M'. This proves (a).

20.2. Let V, V' be two simple $\mathbf{E}$-modules. Let $t : V \to V, t' : V' \to V'$ be $\bar{\mathbf{Q}}_l$-linear invertible maps. Let $N = |\Gamma|^{-1} \sum_{w\in\Gamma} \mathrm{tr}(b_w t, V)\mathrm{tr}(t'^{-1} b_w^{-1}, V')$. We show:

(a)$N = 0$ *if* V, V' *are not isomorphic* $\mathbf{E}$*-modules and* $N = 1$ *if* $V = V', t = t'$.
In the case where $\lambda(w,y) = 1$ for all w, y this is just Schur's orthogonality formula. The proof in the general case is similar. Let $(e_i)_{i\in I}$ be a basis of V and let $(e'_h)_{h\in I'}$ be a basis of V'. For $w \in \Gamma$ we define $\alpha_{ij}^w, \beta_{hk}^w \in \bar{\mathbf{Q}}_l$ by $b_w(e_i) = \sum_{j\in I} \alpha_{ij}^w e_j$, $b_w^{-1}(e'_h) = \sum_{k\in I'} \beta_{hk}^w e'_k$. Define $\xi_{ij}, \zeta_{hk} \in \bar{\mathbf{Q}}_l$ by $t(e_i) = \sum_{j\in I} \xi_{ij} e_j$, $t'^{-1}(e'_h) = \sum_{k\in I'} \zeta_{hk} e'_k$. We have

(b) $$N = |\Gamma|^{-1} \sum_{w\in\Gamma} \sum_{i,j,k,h} \xi_{ij}\alpha_{ji}^w \zeta_{hk}\beta_{kh}^w.$$

For a $\bar{\mathbf{Q}}_l$-linear map $f : V \to V'$ we define $\tilde{f} : V \to V'$ by $\tilde{f}(v) = \sum_{w\in\Gamma} b_w^{-1} f(b_w v)$. As in the proof of 20.1(a) we see that $\tilde{f}$ is $\mathbf{E}$-linear. For $i \in I, h \in I'$ define a linear map $f : V \to V'$ by $f(e_j) = \delta_{ij} e'_h$ for all j. We have

$$\tilde{f}(e_u) = \sum_w b_w^{-1} f(b_w e_u) = \sum_w b_w^{-1} f(\sum_j \alpha_{uj}^w e_j) = \sum_w b_w^{-1} \sum_j \alpha_{uj}^w \delta_{ij} e'_h$$
$$= \sum_w \alpha_{ui}^w b_w^{-1} e'_h = \sum_{w,k} \alpha_{ui}^w \beta_{hk}^w e'_k.$$

If V, V' are not isomorphic $\mathbf{E}$-modules then, by Schur's lemma, we have $\tilde{f} = 0$ hence $\sum_w \alpha_{ui}^w \beta_{hk}^w = 0$ for any u, i, h, k and $N = 0$ as desired. Assume now that $V = V', t = t'$. We may assume that $I = I'$, $e_i = e'_i$. If $f, \tilde{f}$ are as above then, by Schur's lemma, $\tilde{f}$ is a multiple of 1. Since $\mathrm{tr}(\tilde{f}, V) = |\Gamma|\mathrm{tr}(f, V) = |\Gamma|\mathrm{tr}(f, V) = \delta_{ih}|\Gamma|$ we have $\tilde{f} = \delta_{ih}|\Gamma| n^{-1} 1$ where $n = \dim V$. Hence $\sum_w \alpha_{ui}^w \beta_{hk}^w = \delta_{ih}\delta_{uk}|\Gamma| n^{-1}$ for any u, i, h, k and

$$N = \sum_{i,j,k,h} \xi_{ij}\delta_{ik}\delta_{jh}\zeta_{hk} n^{-1} = \sum_{i,j} \xi_{ij}\zeta_{ji} n^{-1} = \sum_i n^{-1} = 1,$$

as desired.

20.3. Since $\mathbf{E}$ is semisimple, we have an algebra isomorphism

(a) $\mathbf{E} \longrightarrow \oplus_{i=1}^{r'} \mathrm{End}(V_i)$,

$e \mapsto (f_i^e)$ where $f_i^e : V_i \longrightarrow V_i$ takes v to ev; here $V_i, (i \in [1, r'])$ is a set of representatives for the isomorphism classes of simple $\mathbf{E}$-modules.

Let $\iota : \mathbf{E} \longrightarrow \mathbf{E}$ be an automorphism of the algebra $\mathbf{E}$. We may assume that for $i \in [1, r]$ the following property holds:

$(*)$ *there exists a $\bar{\mathbf{Q}}_l$-linear isomorphism $\iota_i : V_i \longrightarrow V_i$ such that $\iota_i(ev) = \iota(e)\iota_i(v)$ for all $e \in \mathbf{E}, v \in V$*

and that for $i > r$ this property does not hold. We choose for each $i \in [1, r]$ an isomorphism $\iota_i : V_i \longrightarrow V_i$ as above (it is unique up to a non-zero scalar). We show that, for any w, w' in Γ,

(b) $\sum_{i=1}^r \mathrm{tr}(b_w \iota_i, V_i)\mathrm{tr}(\iota_i^{-1} b_{w'}^{-1}, V_i)$ *is equal to the trace of the linear map* $\kappa : \mathbf{E} \longrightarrow \mathbf{E}, e \mapsto b_{w'}^{-1}\iota^{-1}(e)b_w$.

Let $\tau : \oplus_{i=1}^{r'} \mathrm{End}(V_i) \longrightarrow \oplus_{i=1}^{r'} \mathrm{End}(V_i)$ be the linear map which corresponds to $\kappa : \mathbf{E} \longrightarrow \mathbf{E}$ under the isomorphism (a). For $i \in [1, r]$, τ restricts to a linear map $\tau_i : \mathrm{End}(V_i) \longrightarrow \mathrm{End}(V_i)$, while for $i > r$, τ maps the summand $\mathrm{End}(V_i)$ to a different summand. Hence $\mathrm{tr}(\kappa, \mathbf{E}) = \sum_{i=1}^r \mathrm{tr}(\tau_i, \mathrm{End}(V_i))$. For $i \in [1, k]$, τ_i takes $f \in \mathrm{End}(V_i)$ to $v \mapsto b_{w'}^{-1}\iota_i^{-1}(f(\iota_i(b_w v)))$ hence

$\mathrm{tr}(\tau_i, \mathrm{End}(V_i) = \mathrm{tr}(b_{w'}^{-1}\iota_i^{-1}, V_i)\mathrm{tr}(\iota_i b_w, V_i)$;

(b) follows.

20.4. We now assume that $\iota : \mathbf{E} \longrightarrow \mathbf{E}$ in 20.3 satisfies $\iota(\mathbf{E}_w) = \mathbf{E}_{F(w)}$ for all w, where $F : \Gamma \longrightarrow \Gamma$ is a group isomorphism. For $x \in \Gamma$, let $\Gamma_x = \{y \in \Gamma; F^{-1}(y)xy^{-1} = x\}$; we define $\gamma_x : \Gamma_x \longrightarrow \bar{\mathbf{Q}}_l^*$ by $\iota^{-1}(b_y)b_x = \gamma_x(y)b_x b_y$. We show that γ_x is a group homomorphism. Let $z, z' \in \Gamma_x$. We have $b_z b_{z'} = u b_{zz'}$ with $u \in \bar{\mathbf{Q}}_l^*$. We have

$$\begin{aligned}
&\iota^{-1}(b_{zz'})b_x = u^{-1}\iota^{-1}(b_z b_{z'})b_x = u^{-1}\iota^{-1}(b_z)\gamma_x(z')b_x b_{z'} \\
&= u^{-1}\gamma_x(z')\iota^{-1}(b_z)b_x b_{z'} = u^{-1}\gamma_x(z')\gamma_x(z)b_x b_z b_{z'} = \gamma_x(z')\gamma_x(z)b_x b_{zz'}
\end{aligned}$$

hence $\gamma_x(zz') = \gamma_x(z')\gamma_x(z)$, as desired.

An element $x \in \Gamma$ is said to be *effective* if γ_x is identically 1. For $x, y \in \Gamma, z \in \Gamma_x$ we have $yzy^{-1} \in \Gamma_{F^{-1}(y)xy^{-1}}$ and $\gamma_x(z) = \gamma_{F^{-1}(y)xy^{-1}}(yzy^{-1})$. It follows that the set of effective elements in Γ is a union of F-twisted conjugacy classes. We say that an F-twisted conjugacy class in Γ is effective if some/any element of it is effective.

(a) *If an F-twisted conjugacy class C is not effective then for $i \in [1, r]$ and $x \in C$ we have* $\mathrm{tr}(\iota_i b_x, V_i) = 0$.

Indeed, we can find $y \in \Gamma_x$ such that $\gamma_x(y) \neq 1$. We have

$$\begin{aligned}
&\mathrm{tr}(\iota_i b_x, V_i) = \mathrm{tr}(b_y^{-1}\iota_i b_x b_y, V_i) = \gamma_x(y)^{-1}\mathrm{tr}(b_y^{-1}\iota_i\iota^{-1}(b_y)b_x, V_i) \\
&= \gamma_x(y)^{-1}\mathrm{tr}(\iota_i b_x, V_i).
\end{aligned}$$

Thus $(1 - \gamma_x(y)^{-1})\mathrm{tr}(\iota_i b_x, V_i) = 0$ and $\mathrm{tr}(\iota_i b_x, V_i) = 0$ as claimed.

(b) *If* $i,j \in [1,r]$ *then* $x \mapsto \mathrm{tr}(b_x\iota_i, V_i)\mathrm{tr}(\iota_j^{-1}b_x^{-1}, V_j)$ *is constant on any* F*-twisted conjugacy class.*
Indeed let $y \in \Gamma$. We have $b(F^{-1}(y)xy^{-1}) = c\iota^{-1}(b_y)b_xb_y^{-1}$ for some $c \in \bar{\mathbf{Q}}_l^*$. Hence

$$\begin{aligned}&\mathrm{tr}(b_{F^{-1}(y)xy^{-1}}\iota_i, V_i)\mathrm{tr}(\iota_j^{-1}b_{F^{-1}(y)xy^{-1}}^{-1}, V_j)\\ &= \mathrm{tr}(c\iota^{-1}(b_y)b_xb_y^{-1}\iota_i, V_i)\mathrm{tr}(\iota_j^{-1}c^{-1}b_yb_x^{-1}\iota^{-1}(b_y)^{-1}, V_j)\\ &= \mathrm{tr}(\iota^{-1}(b_y)b_x\iota_i\iota^{-1}(b_y)^{-1}, V_i)\mathrm{tr}(\iota^{-1}(b_y)\iota_j^{-1}b_x^{-1}\iota^{-1}(b_y)^{-1}, V_j)\\ &= \mathrm{tr}(b_x\iota_i, V_i)\mathrm{tr}(\iota_j^{-1}b_x^{-1}, V_j)\end{aligned}$$

and (b) follows.

Let $\bar{\Gamma}$ be a set of representatives for the effective F-twisted conjugacy classes in Γ. We rewrite 20.2(a) for $V = V_i, V' = V_j, t = \iota_i, t' = \iota_j$ where $i,j \in [1,r]$, taking into account (a),(b):

(c)
$$\sum_{x \in \bar{\Gamma}} |\Gamma_x|^{-1}\mathrm{tr}(b_x\iota_i, V_i)\mathrm{tr}(\iota_j^{-1}b_x^{-1}, V_j) = \delta_{ij}.$$

We rewrite 20.3(b) for $w, w' \in \bar{\Gamma}$ as follows

(d)
$$\sum_{i=1}^{r} \mathrm{tr}(b_w\iota_i, V_i)\mathrm{tr}(\iota_i^{-1}b_{w'}^{-1}, V_i) = \delta_{w,w'}|\Gamma_w|.$$

Indeed, it is enough to show that the trace of κ in 20.3(b) is in our case $\delta_{w,w'}|\Gamma_w|$. Now that trace is $\sum_{y \in \Gamma; w'^{-1}F^{-1}(y)w = y} c_y$ where $c_y \in \bar{\mathbf{Q}}_l^*$ is defined by $b_{w'}^{-1}\iota^{-1}(b_y)b_w = c_yb_y$. If $w' \neq w$, the last sum is empty so its value is 0. If $w' = w$, the last sum is $\sum_{y \in \Gamma_w} \gamma_w(y)$ and this equals $|\Gamma_w|$ since γ_w is identically 1.

(e) *The matrix* $(\mathrm{tr}(b_x\iota_i, V_i))_{i \in [1,r], x \in \bar{\Gamma}}$ *is square and invertible.*
Indeed, from (c),(d) we see that this matrix has a left inverse and a right inverse. The same argument shows that

(f) *the matrix* $(\mathrm{tr}(\iota_i^{-1}b_x^{-1}, V_i))_{i \in [1,r], x \in \bar{\Gamma}}$ *is square and invertible.*
In particular,

(g) $|\bar{\Gamma}| = r$.

21. Bases

21.1. If L is a Levi of a parabolic of G^0, let $N_G^\bullet L$ be the set of all $g \in N_GL$ such that for some parabolic P of G^0 with Levi L we have $g \in N_GP$. Then $N_G^\bullet L$ is a union of connected components of N_GL.

Let $\mathfrak{A}_G$ be the set of all pairs $(L, \mathfrak{c})$ where L is a Levi subgroup of some parabolic of G^0 and $\mathfrak{c}$ is a unipotent cuspidal L-conjugacy class in $N_G^\bullet L$.

Let G_{un} be the set of unipotent elements in G.

21.2. Let L be a Levi of a parabolic P of G^0 and let $g \in \tilde{L} = N_G L \cap N_G P$. Then

(a) *g is quasi-semisimple in G if and only if g is quasi-semisimple in $\tilde{L}$.*

See [DM, 1.10].

21.3. For a fixed $g \in G$, let $\mathfrak{R}$ be the set of all $\underline{L}$ such that $\underline{L}$ is a Levi of a parabolic of G^0, $g \in N^\bullet_G \underline{L}$ and g is isolated in $N_G \underline{L}$; let $\mathfrak{R}'$ be the set of all L such that L is a Levi of a parabolic of $Z_G(g_s)^0$ and $g \in N^\bullet_{Z_G(g_s)} L$. We show:

(a) *$\underline{L} \mapsto a(\underline{L}) = \underline{L} \cap Z_G(g_s)^0, L \mapsto b(L) = Z_{G^0}((\mathcal{Z}^0_L \cap Z_G(g))^0)$ define inverse bijections $\mathfrak{R} \leftrightarrow \mathfrak{R}'$.*

Let $L \in \mathfrak{R}'$. Set $\underline{L} = b(L)$. Then $\underline{L}$ is a Levi of a parabolic of G^0. Clearly, $g \in N_G \underline{L}$. If $\chi : \mathbf{k}^* \to (\mathcal{Z}^0_L \cap Z_G(g))^0$ is general enough, then

$Z_{Z_G(g_s)^0}(\chi(\mathbf{k}^*)) = Z_{Z_G(g_s)^0}(\mathcal{Z}^0_L \cap Z_G(g))^0 = L$ (see 1.10) and $Z_{G^0}(\chi(\mathbf{k}^*)) = Z_{G^0}(\mathcal{Z}^0_L \cap Z_G(g))^0 = \underline{L}$.

Then $\underline{L}$ is a Levi of the parabolic $P = P_\chi$ (see 1.16). We have $g\chi(t)g^{-1} = \chi(t)$ for all $t \in \mathbf{k}^*$, hence $gPg^{-1} = P$. Thus $g \in N^\bullet_G \underline{L}$. We have

$$\underline{L} \cap Z_G(g_s)^0 = Z_{G^0}(\chi(\mathbf{k}^*)) \cap Z_G(g_s)^0 = Z_{Z_G(g_s)^0}(\chi(\mathbf{k}^*)) = L.$$

From $L = Z_G(g_s)^0 \cap \underline{L}$ we see that $L = Z_{\underline{L}}(g_s)^0$. Hence $\mathcal{Z}^0_{Z_{\underline{L}}(g_s)^0} = \mathcal{Z}^0_L$,

$$T_{N_G \underline{L}}(g) = (\mathcal{Z}^0_{Z_{\underline{L}}(g_s)^0} \cap Z_G(g))^0 = (\mathcal{Z}^0_L \cap Z_G(g))^0 \subset \mathcal{Z}^0_{\underline{L}}.$$

By 2.2(ii), we see that g is isolated in $N_G \underline{L}$. Hence $\underline{L} \in \mathfrak{R}$ and b is well defined.

Conversely, let $\underline{L} \in \mathfrak{R}$. Set $L = a(\underline{L})$. Let P be a parabolic of G^0 with Levi $\underline{L}$ such that $g \in N_G \underline{L} \cap N_G P$. Let $Q = P \cap Z_G(g_s)^0$. By 1.12, Q is a parabolic of $Z_G(g_s)^0$ with Levi L. Clearly, $g \in Z_G(g_s)$ normalizes L and Q. Hence $L \in \mathfrak{R}'$ and a is well defined. Since g is isolated in $N_G \underline{L}$, we have (see 2.2(iii)):

$$(\mathcal{Z}^0_{\underline{L}} \cap Z_G(g))^0 = T_{N_G \underline{L}}(g) = (\mathcal{Z}^0_{Z_{\underline{L}}(g_s)^0} \cap Z_G(g))^0 = (\mathcal{Z}^0_L \cap Z_G(g))^0$$

hence (using 1.10):

$$\underline{L} = Z_{G^0}((\mathcal{Z}^0_{\underline{L}} \cap Z_G(g))^0) = Z_{G^0}((\mathcal{Z}^0_L \cap Z_G(g))^0) = b(L) = ba(\underline{L}).$$

Thus, $ba = 1$. As we have seen above, for $L \in \mathfrak{R}'$ we have $ab(L) = L$. This proves (a).

21.4. In the remainder of this section we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$ and that G has a fixed $\mathbf{F}_q$-rational structure with Frobenius map $F : G \to G$.

Let $\mathcal{R}$ be the set of all triples $(g, \underline{L}, \mathbf{c})$ where $g \in G^F$ is quasi-semisimple, $\underline{L}$ is a Levi of a parabolic of G^0, $F(\underline{L}) = \underline{L}$, $\mathbf{c}$ is a a cuspidal $\underline{L}$-conjugacy class in $N_G \underline{L}$ such that $F(\mathbf{c}) = \mathbf{c}$, $\mathbf{c} \subset N^\bullet_G \underline{L}$ and $g \in \sigma_{N_G \underline{L}}(\mathbf{c})$.

Let $\mathcal{R}'$ be the set of all triples $(g, L, \mathfrak{c})$ where $g \in G^F$ is quasi-semisimple, L is a Levi of a parabolic of $Z_G(g_s)^0$, $F(L) = L$, and $\mathfrak{c}$ is a unipotent cuspidal L-conjugacy class in $N_{Z_G(g_s)}L$ such that $F(\mathfrak{c}) = \mathfrak{c}$, $\mathfrak{c} \subset g_u L$, $\mathfrak{c} \subset N^\bullet_{Z_G(g_s)}L$ (we have automatically $g \in N_{Z_G(g_s)}L$). Define $\tilde{a} : \mathcal{R} \to \mathcal{R}'$ by

(a) $(g, \underline{L}, \mathbf{c}) \mapsto (g, L, \mathfrak{c}), L = a(\underline{L}), \mathfrak{c} = \{u \in g_u L; u \text{ unipotent}, g_s u \in \mathbf{c}\}$.

To see that $(g, L, \mathfrak{c}) \in \mathcal{R}'$ we note that $\mathfrak{c}$ is a single L-conjugacy class, by 17.13 applied to $N_G\underline{L}$ instead of G; also from 20.7(a) we have $g \in N^\bullet_{Z_G(g_s)}L, \mathfrak{c} \subset g_u L$ hence $\mathfrak{c} \subset N^\bullet_{Z_G(g_s)}L$.

Define $\tilde{b} : \mathcal{R}' \to \mathcal{R}$ by

(b) $(g, L, \mathfrak{c}) \mapsto (g, \underline{L}, \mathbf{c}), \underline{L} = b(L), \mathbf{c} = \cup_{l \in \underline{L}} l g_s \mathfrak{c} l^{-1}$.

To see that $(g, \underline{L}, \mathbf{c}) \in \mathcal{R}$ we note that there exists $x \in \mathbf{c}$ with $x_s = g_s, x_u \in Z_{\underline{L}}(g_s)^0 g_u$ and that g is isolated in $N_G\underline{L}$, $g \in N^\bullet_G\underline{L}$ (see 20.7(a)); it follows that $g \in \sigma_{N_G\underline{L}}(\mathbf{c})$ and that $\mathbf{c}$ is isolated in $N_G\underline{L}$, $\mathbf{c} \in N^\bullet_G\underline{L}$ (we apply Lemma 2.5 with $N_G\underline{L}$ instead of G to g and x as above).

From the definition we see that

(c) *the maps (a),(b) are inverse bijections* $\mathcal{R} \leftrightarrow \mathcal{R}'$.

21.5. Assume that $(g, \underline{L}, \mathbf{c}) \in \mathcal{R}, (g, L, \mathfrak{c}) \in \mathcal{R}$ correspond to each other under the bijections 20.8(c). Let $\langle g \rangle = \cup_{l \in \underline{L}^F} l g l^{-1}$. Let

$\mathcal{G} = \{y \in Z_{G^0}(g_s)^F; yLy^{-1} = L, y\mathfrak{c}y^{-1} = \mathfrak{c}\}$,

$\mathcal{G}' = \{y \in G^{0F}, y\underline{L}y^{-1} = \underline{L}, y\mathbf{c}y^{-1} = \mathbf{c}, y\langle g \rangle y^{-1} = \langle g \rangle\}$.

We show:

(a) $\mathcal{G}' = \underline{L}^F \mathcal{G}$.

Let $y \in \mathcal{G}$. Since $\mathfrak{c} \subset g_u L$ and $yLy^{-1} = L, y\mathfrak{c}y^{-1} = \mathfrak{c}$, we have $\mathfrak{c} \subset y g_u y^{-1} L$. Hence $y g_u y^{-1}, g_u$ are two F-stable, unipotent quasi-semisimple elements of $N_G L$ in the same connected component of $N_G L$. Using 19.11 we have $y g_u y^{-1} = l g_u l^{-1}$ for some $l \in L^F$. We have $\underline{L} = Z_{G^0}((\mathcal{Z}^0_L \cap Z_G(g))^0) = Z_{G^0}((\mathcal{Z}^0_L \cap Z_G(g_u))^0)$. Hence

$$
\begin{aligned}
y\underline{L}y^{-1} &= Z_{G^0}((\mathcal{Z}^0_{yLy^{-1}} \cap Z_G(y g_u y^{-1}))^0) \\
&= Z_{G^0}((\mathcal{Z}^0_L \cap Z_G(l g_u l^{-1}))^0) = Z_{G^0}((\mathcal{Z}^0_L \cap Z_G(g_u))^0) = \underline{L}.
\end{aligned}
$$

Since $\mathbf{c} = \cup_{l' \in \underline{L}} l' g_s \mathfrak{c} l'^{-1}$ we see that $y\mathbf{c}y^{-1} = \mathbf{c}$. We have $ygy^{-1} = y g_s y^{-1} y g_u y^{-1} = g_s l g_u l^{-1} = l g l^{-1} \in \langle g \rangle$. Hence $y\langle g \rangle y^{-1} = \langle g \rangle$. We see that $y \in \mathcal{G}'$. Thus, $\mathcal{G} \subset \mathcal{G}'$. The inclusion $\underline{L}^F \subset \mathcal{G}'$ is obvious. Hence $\underline{L}^F \mathcal{G} \subset \mathcal{G}'$. Conversely, let $y \in \mathcal{G}'$. Then $y = l'y'$ where $l' \in \underline{L}^F$, $y' \in \mathcal{G}'$, $y'gy'^{-1} = g$. We have $y'\underline{L}y'^{-1} = \underline{L}$ hence $y'Ly'^{-1} = L$. We have $y'\mathbf{c}y'^{-1} = \mathbf{c}$, $y'(g_u L)y'^{-1} = g_u L$ hence $y'\mathfrak{c}y'^{-1} = \mathfrak{c}$. Thus $y' \in \mathcal{G}$ and (a) holds.

21.6. Let $(L, S) \in \mathbf{A}$ and let $\mathcal{E} \in \mathcal{S}(S)$ be an irreducible cuspidal local system on S (relative to $N_G L$). Let $Y = Y_{L,S}, \tilde{Y} = \tilde{Y}_{L,S}, \pi : \tilde{Y} \to Y$ be as in 3.13. Let $\tilde{\mathcal{E}}$ be the local system on $\tilde{Y}$ defined in 5.6. Let $\tilde{\Gamma} = \{n \in N_{G^0}L; nSn^{-1} = S, Ad(n)^*\mathcal{E} \cong \mathcal{E}\}$ and let $\Gamma = \tilde{\Gamma}/L$. Let $\bar{Y}$ be the closure of Y in G and let $\mathfrak{K} = IC(\bar{Y}, \pi_!\tilde{\mathcal{E}})$. Assume that $FL = L, FS = S$, $F^*\mathcal{E} \cong \mathcal{E}$. Now $F : G \to G$ induces isomorphisms $F : \tilde{\Gamma} \to \tilde{\Gamma}$, $F : \Gamma \to \Gamma$.

For any $n \in \tilde{\Gamma}^F$ there is a well defined element $\eta(n) \in \bar{\mathbf{Q}}_l^*$ such that the following holds: if $\alpha : \mathrm{Ad}(n)^*\mathcal{E} \to \mathcal{E}, \epsilon : F^*\mathcal{E} \to \mathcal{E}$ are isomorphisms, then for any $g \in S$, the composition $\mathcal{E}_{nF(g)n^{-1}} \xrightarrow{\alpha} \mathcal{E}_{F(g)} \xrightarrow{\epsilon} \mathcal{E}_g$ is $\eta(n)$ times the composition $\mathcal{E}_{nF(g)n^{-1}} \xrightarrow{\epsilon} \mathcal{E}_{ngn^{-1}} \xrightarrow{\alpha} \mathcal{E}_g$. (This follows from the irreducibility of $\mathcal{E}$ and Schur's lemma.) Clearly, $\eta(n)$ is independent of the choice of α, ϵ. It follows that, if $g \in S^F$ then the composition $\mathcal{E}_{ngn^{-1}} \xrightarrow{\alpha} \mathcal{E}_g \xrightarrow{\epsilon} \mathcal{E}_g$ is $\eta(n)$ times the composition $\mathcal{E}_{ngn^{-1}} \xrightarrow{\epsilon} \mathcal{E}_{ngn^{-1}} \xrightarrow{\alpha} \mathcal{E}_g$. Hence $\chi_{\mathcal{E},\epsilon}(g) = \eta(n)\chi_{\mathcal{E},\epsilon}(ngn^{-1})$. This property characterizes $\eta(n)$ since $S^F \neq \emptyset$. Since $\chi_{\mathcal{E},\epsilon}$ is constant on L^F-conjugacy classes in S^F it follows that η induces a homomorphism $\tilde{\Gamma}^F/L^F = (\tilde{\Gamma}/L)^F = \Gamma^F \to \bar{\mathbf{Q}}_l^*$. We say that $(L, S, \mathcal{E})$ is *effective* if the associated homomorphism $\eta : \Gamma^F \to \bar{\mathbf{Q}}_l^*$ is identically 1. (In this case we have $\chi_{\mathcal{E},\epsilon}(ngn^{-1}) = \chi_{\mathcal{E},\epsilon}(g)$ for any $g \in S^F, n \in \tilde{\Gamma}^F$.)

We fix an isomorphism $\epsilon : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. Now F induces Frobenius maps on $Y, \tilde{Y}, \bar{Y}$. Also, ϵ induces an isomorphism $F^*\tilde{\mathcal{E}} \xrightarrow{\sim} \tilde{\mathcal{E}}$ of local systems on $\tilde{Y}$, an isomorphism $F^*\pi_!\tilde{\mathcal{E}} \xrightarrow{\sim} \pi_!\tilde{\mathcal{E}}$ of local systems on Y and an isomorphism $\phi : F^*\mathfrak{K} \xrightarrow{\sim} \mathfrak{K}$ in $\mathcal{D}(\bar{Y})$. As in 7.10, let $\mathbf{E}$ be the algebra of endomorphisms of the local system $\pi_!\tilde{\mathcal{E}}$ on Y. As in 7.10(a) we have a canonical decomposition $\mathbf{E} = \oplus_{w \in \Gamma}\mathbf{E}_w$ where $\mathbf{E}_w$ is a one dimensional subspace of $\mathbf{E}$. From the definitions we see that for $w, y \in \Gamma$ we have $\mathbf{E}_w\mathbf{E}_y = \mathbf{E}_{wy}$. Let b_w be a basis element of $\mathbf{E}_w$. Then $\mathbf{E}, \mathbf{E}_w, b_w$ are as in 20.1. As in 20.3, let $V_i, (i \in [1, r'])$ be a set of representatives for the isomorphism classes of simple $\mathbf{E}$-modules. We have canonically $\mathbf{E} = \mathrm{End}(\mathfrak{K})$. For $i \in [1, r']$ let $(\pi_!\tilde{\mathcal{E}})_i = \mathrm{Hom}_{\mathbf{E}}(V_i, \pi_!\tilde{\mathcal{E}})$, $\mathfrak{K}_i = \mathrm{Hom}_{\mathbf{E}}(V_i, \mathfrak{K})$. Then $(\pi_!\tilde{\mathcal{E}})_i$ is an irreducible local system on Y and $\mathfrak{K}_i = IC(\bar{Y}, (\pi_!\tilde{\mathcal{E}})_i)$. Also, $(\pi_!\tilde{\mathcal{E}})_i \not\cong (\pi_!\tilde{\mathcal{E}})_{i'}$ for $i \neq i'$. We have canonically $\pi_!\tilde{\mathcal{E}} = \oplus_{i \in [1,r']}V_i \otimes (\pi_!\tilde{\mathcal{E}})_i$ and $\mathfrak{K} = \oplus_{i \in [1,r']}V_i \otimes \mathfrak{K}_i$. For $f \in \mathrm{Hom}(\pi_!\tilde{\mathcal{E}}, \pi_!\tilde{\mathcal{E}})$ we have

$$F^*(f) \in \mathrm{Hom}(F^*\pi_!\tilde{\mathcal{E}}, F^*\pi_!\tilde{\mathcal{E}}) = \mathrm{Hom}(\pi_!F^*\tilde{\mathcal{E}}, \pi_!F^*\tilde{\mathcal{E}}) = \mathrm{Hom}(\pi_!\tilde{\mathcal{E}}, \pi_!\tilde{\mathcal{E}})$$

where the last equality is obtained by using twice the isomorphism $F^*\tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ as above. Hence we have a map $\iota : \mathbf{E} \to \mathbf{E}, f \mapsto F^*(f)$ which is an algebra isomorphism; it carries $\mathbf{E}_w$ onto $\mathbf{E}_{F(w)}$ for any $w \in \Gamma$. As in 20.3 we may assume that for $i \in [1, r]$, property 20.3(∗) holds and that for $i > r$ that property does not hold. For $i \in [1, r]$ we choose an isomorphism $\iota_i : V_i \to V_i$ as in 20.3(∗). For any $w \in \Gamma$, the isomorphism $b_w\phi : F^*\mathfrak{K} \xrightarrow{\sim} \mathfrak{K}$ corresponds under $\mathfrak{K} = \oplus_{i \in [1,r']}V_i \otimes \mathfrak{K}_i$ to an isomorphism

(a) $\oplus_{i \in [1,r']}V_i \otimes F^*\mathfrak{K}_i \to \oplus_{i \in [1,r']}V_i \otimes \mathfrak{K}_i$

which is an isomorphism of the summand $V_i \otimes F^*\mathfrak{K}_i$ onto a summand $V_{i'} \otimes \mathfrak{K}_{i'}$ where $i' = i$ for $i \in [1, r]$ and $i' \neq i$ for $i > r$; moreover, the restriction of (a) to $V_i \otimes F^*\mathfrak{K}_i, i \in [1, r]$, is of the form $b_w\iota_i \otimes \phi_i$ where $\phi_i : F^*\mathfrak{K}_i \xrightarrow{\sim} \mathfrak{K}_i$ is an isomorphism independent of w. (Note that $F^*\mathfrak{K}_i \not\cong \mathfrak{K}_i$ for $i > r$.) Taking induced maps on stalks and taking traces we obtain for any $j \in \mathbf{Z}, g \in \bar{Y}^F$:

$$\mathrm{tr}(b_w\phi, \mathcal{H}_g^j\mathfrak{K}) = \sum_{i \in [1,r]} \mathrm{tr}(b_w\iota_i, V_i)\mathrm{tr}(\phi_i, \mathcal{H}^j\mathfrak{K}_i).$$

Taking alternating sum over j we obtain

(b) $$\chi_{\mathfrak{K},b_w\phi}=\sum_{i\in[1,r]}\mathrm{tr}(b_w\iota_i,V_i)\chi_{\mathfrak{K}_i,\phi_i}.$$

We choose an element $g_w\in G^0$ such that $g_w^{-1}F(g_w)=n_w$, a representative of w in $\tilde\Gamma$. We set $L^w=g_wLg_w^{-1}, S^w=g_wSg_w^{-1}$, $\mathcal{E}^w=\mathrm{Ad}(g_w^{-1})^*\mathcal{E}$ (a local system on S^w). Then $F(L^w)=L^w$ and $F(S^w)=S^w$. We define an isomorphism $\epsilon^w:F^*\mathcal{E}^w\xrightarrow{\sim}\mathcal{E}^w$ in terms of $\epsilon:F^*\mathcal{E}\xrightarrow{\sim}\mathcal{E}$ and b_w as follows. By definition, b_w defines for each $g\in S$ an isomorphism of stalks $\mathcal{E}_{n_wgn_w^{-1}}\xrightarrow{\sim}\mathcal{E}_g$; hence it defines for any $g'\in S^w$ an isomorphism $\mathcal{E}_{n_wF(g_w)^{-1}F(g')F(g_w)n_w^{-1}}\xrightarrow{\sim}\mathcal{E}_{F(g_w)^{-1}F(g')F(g_w)}$ or equivalently $\mathcal{E}_{g_w^{-1}F(g')g_w}\xrightarrow{\sim}\mathcal{E}_{F(g_w^{-1}g'g_w)}$. Composing with $\epsilon:\mathcal{E}_{F(g_w^{-1}g'g_w)}\xrightarrow{\sim}\mathcal{E}_{g_w^{-1}g'g_w}$ we obtain $\mathcal{E}_{g_w^{-1}F(g')g_w}\xrightarrow{\sim}\mathcal{E}_{g_w^{-1}g'g_w}$ that is, $\mathcal{E}^w_{F(g')}\xrightarrow{\sim}\mathcal{E}^w_{g'}$ which comes from an isomorphism $\epsilon^w:F^*\mathcal{E}^w\xrightarrow{\sim}\mathcal{E}^w$. We define $\pi^w:\tilde Y^w\to Y^w$, $\tilde{\mathcal{E}}^w,\mathfrak{K}^w$, $\phi^w:F^*\mathfrak{K}^w\xrightarrow{\sim}\mathfrak{K}^w$ in terms of $L^w,S^w,\mathcal{E}^w,\epsilon^w$ in the same way as $\pi:\tilde Y\to Y$, $\tilde{\mathcal{E}},\mathfrak{K}$, $\phi:F^*\mathfrak{K}\xrightarrow{\sim}\mathfrak{K}$ were defined in terms of $L,S,\mathcal{E},\epsilon$. We have $Y^w=Y$ and the map $(g,xL)\mapsto(g,xg_w^{-1}L^w)$ is an isomorphism $\mu:\tilde Y\to\tilde Y^w$ commuting with the projections π,π^w onto Y. We have $\mu^*\tilde{\mathcal{E}}^w=\tilde{\mathcal{E}}$ canonically. Hence μ induces an isomorphism $\pi_!\tilde{\mathcal{E}}\xrightarrow{\sim}\pi^w_!\tilde{\mathcal{E}}^w$ hence an isomorphism $\mu':\mathfrak{K}\xrightarrow{\sim}\mathfrak{K}^w$. From the definitions we see that the compositions $F^*\mathfrak{K}\xrightarrow{F^*\mu'}F^*\mathfrak{K}^w\xrightarrow{\phi^w}\mathfrak{K}^w$, $F^*\mathfrak{K}\xrightarrow{b_w\phi}\mathfrak{K}\xrightarrow{\mu'}\mathfrak{K}^w$ coincide. Hence for $j\in\mathbf{Z},g\in\bar Y^F$ we have $\mathrm{tr}(b_w\phi,\mathcal{H}^j_g\mathfrak{K})=\mathrm{tr}(\phi^w,\mathcal{H}^j_g\mathfrak{K}^w)$. Taking alternating sum over j gives $\chi_{\mathfrak{K},b_w\phi}=\chi_{\mathfrak{K}^w,\phi^w}$. Introducing this in (b) we obtain

(c) $$\chi_{\mathfrak{K}^w,\phi^w}=\sum_{i\in[1,r]}\mathrm{tr}(b_w\iota_i,V_i)\chi_{\mathfrak{K}_i,\phi_i}.$$

Using (c) for w running through $\bar\Gamma$ (a set of representatives for the effective F-twisted conjugacy classes in Γ) and 20.4(e),(g), we see that

(d) *the functions* $(\chi_{\mathfrak{K}^w,\phi^w})_{w\in\bar\Gamma}$ *span the same vector space as the functions* $(\chi_{\mathfrak{K}_i,\phi_i})_{i\in[1,r]}$*; moreover* $|\bar\Gamma|=r$.

From the definitions we see that

(e) $(L^w,S^w,\mathcal{E}^w)$ *is effective if and only if* $w\in\Gamma$ *is effective.*

We show that,

(f) *if* $x\in\Gamma$ *is not effective, then* $\chi_{\mathfrak{K}^x,\phi^x}=0$.

It is enough to show that, for any $j\in\mathbf{Z},g\in\bar Y^F$ we have $\mathrm{tr}(b_x\phi,\mathcal{H}^j_g\mathfrak{K})=0$. The proof is a repetition of that of 20.4(a). We can find $y\in\Gamma_x$ such that $\gamma_x(y)\ne1$. (Notation of 20.4.) We have

$$\mathrm{tr}(\phi b_x,\mathcal{H}^j_g\mathfrak{K})=\mathrm{tr}(b_y^{-1}\phi b_xb_y,\mathcal{H}^j_g\mathfrak{K})=\gamma_x(y)^{-1}\mathrm{tr}(b_y^{-1}\phi\iota^{-1}(b_y)b_x,\mathcal{H}^j_g\mathfrak{K})$$
$$=\gamma_x(y)^{-1}\mathrm{tr}(\phi b_x,\mathcal{H}^j_g\mathfrak{K}).$$

Thus $(1-\gamma_x(y)^{-1})\mathrm{tr}(\phi b_x,\mathcal{H}^j_g\mathfrak{K})=0$ and $\mathrm{tr}(\phi b_x,\mathcal{H}^j_g\mathfrak{K})=0$ as claimed.

21.7. We preserve the setup of 21.6. Let Ξ be the set of all triples $(L', S', [\mathcal{E}'])$ such that

$(L', S') \in \mathbf{A}$,

$[\mathcal{E}']$ is the isomorphism class of an irreducible cuspidal local system $\mathcal{E}' \in \mathcal{S}(S')$ (relative to $N_G L'$),

there exists $g \in G^0$ such that $gLg^{-1} = L', gSg^{-1} = S'$ and $\mathrm{Ad}(g^{-1})^* \mathcal{E} \in [\mathcal{E}']$. Note that G^0 acts naturally on Ξ and this action is transitive. The isotropy group of $(L, S, [\mathcal{E}])$ in G^0 is $\tilde{\Gamma}$. Thus we may identify $\Xi = G^0/\tilde{\Gamma}$. Since $\tilde{\Gamma}$ is F-stable, we have an induced Frobenius map $F : \Xi \longrightarrow \Xi$ whose fixed point set consists of all $(L', S', [\mathcal{E}']) \in \Xi$ such that $F(L') = L', F(S') = S'$ and $F^*\mathcal{E}' \cong \mathcal{E}'$. By 19.7(a), the triples $(L^w, S^w, \mathcal{E}^w)$ (where w runs through a set of representatives of the F-twisted conjugacy classes in Γ) form a set of representatives for the G^{0F}-orbits in Ξ^F. Using now 21.6(e) we see that the triples $(L^w, S^w, \mathcal{E}^w)$ (where w runs through $\bar{\Gamma}$) form a set of representatives for the G^{0F}-orbits on the set of effective triples in Ξ^F.

Let $({}^hL, {}^hS, [{}^h\mathcal{E}])_{h \in I}$ be a set of representatives for the G^{0F}-orbits on the set of effective triples in Ξ^F. For each $h \in I$ choose ${}^h\mathcal{E} \in [{}^h\mathcal{E}]$ and an isomorphism ${}^h\epsilon : F^*({}^h\mathcal{E}) \xrightarrow{\sim} {}^h\mathcal{E}$. Define ${}^h\mathfrak{K}$, ${}^h\phi : F^*({}^h\mathfrak{K}) \xrightarrow{\sim} {}^h\mathfrak{K}$ in terms of ${}^hL, {}^hS, {}^h\mathcal{E}, {}^h\epsilon$ in the same way as $\mathfrak{K}$, $\phi : F^*\mathfrak{K} \xrightarrow{\sim} \mathfrak{K}$ were defined in terms of $L, S, \mathcal{E}, \epsilon$. Let $(A_j)_{j \in J}$ be a set of representatives for the isomorphism classes of simple intersection cohomology complexes A on $\bar{Y}$ that are summands of $\mathfrak{K}$ and satisfy $F^*A \cong A$. For each $j \in J$ choose an isomorphism $\psi_j : F^*A_j \xrightarrow{\sim} A_j$. We can now reformulate 21.6(d) as follows.

(a) *the functions* $(\chi_{{}^h\mathfrak{K}, {}^h\phi})_{h \in I}$ *span the same vector space as the functions* $(\chi_{A_j, \psi_j})_{j \in J}$*; moreover,* $|I| = |J|$.

21.8. Let $L, S, \mathcal{E}$ be as in 21.6. Assume that S contains a unipotent L-conjugacy class $\mathfrak{c}$ (necessarily unique hence F-stable), that $\mathcal{E}$ is the inverse image under $S \longrightarrow \mathfrak{c}, g \mapsto g_u$ of an irreducible cuspidal local system $\mathcal{F}$ on $\mathfrak{c}$ and that $\epsilon : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is induced via $S \longrightarrow \mathfrak{c}$ by an isomorphism $\epsilon_0 : F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. We show that

(a) $(L, S, \mathcal{E})$ *is effective.*

Let $\bar{Y}, \mathfrak{K}, \phi : F^*\mathfrak{K} \xrightarrow{\sim} \mathfrak{K}, \mathbf{E}, b_w, \Gamma, F : \Gamma \longrightarrow \Gamma, \iota : \mathbf{E} \longrightarrow \mathbf{E}$ be as in 21.6. By 21.6(e) it is enough to show that $1 \in \Gamma$ is effective (relative to the basis b_w of $\mathbf{E}$ and $F : \Gamma \longrightarrow \Gamma$). We may assume that b_w are chosen as in the proof of Proposition 11.9. We must show that $y \in \Gamma, F(y) = y \implies \iota^{-1}(b_y) = b_y$. Hence it is enough to show that $\iota(b_y) = b_{F(y)}$ for any $y \in \Gamma$. Let $\mathbf{c}$ be the unique unipotent G^0-conjugacy class of G that is open dense in $\bar{Y} \cap G_{un}$ (see 10.3). Let $\mathcal{H} = \mathcal{H}^0\mathfrak{K}|_{\mathbf{c}}$, an irreducible local system on G^0 (see 11.8). The natural action of $\mathbf{E}$ on $\mathfrak{K}$ induces an action of $\mathbf{E}$ on $\mathcal{H}$ in which b_y acts as the identity map (by the choice of b_y).

Let $f \in \mathrm{Hom}(\mathfrak{K}, \mathfrak{K}) = \mathbf{E}$. The commutative diagram

$$\begin{array}{ccc} F^*\mathfrak{K} & \xrightarrow{F^*f} & F^*\mathfrak{K} \\ \phi \downarrow & & \phi \downarrow \\ \mathfrak{K} & \xrightarrow{\iota(f)} & \mathfrak{K} \end{array}$$

(which comes from the definition of ι) induces a commutative diagram

$$\begin{array}{ccc} F^*\mathcal{H} & \xrightarrow{F^*f} & F^*\mathcal{H} \\ \phi \downarrow & & \phi \downarrow \\ \mathcal{H} & \xrightarrow{\iota(f)} & \mathcal{H} \end{array}$$

If $f = b_y$ then $f : \mathcal{H} \to \mathcal{H}$ is the identity map hence $F^*f : F^*\mathcal{H} \to F^*\mathcal{H}$ is the identity map. Then the last commutative diagram shows that $\iota(f) : \mathcal{H} \to \mathcal{H}$ is the identity map. Since $\iota(b_y)$ is a scalar multiple of $b_{F(y)}$ and $b_{F(y)}$ acts on $\mathcal{H}$ as the identity map, it follows that $\iota(b_y) = b_{F(y)}$, as required.

21.9. Let $\mathcal{V}$ be the vector space of functions $G^F_{un} \to \bar{\mathbf{Q}}_l$ that are constant on G^{0F}-conjugacy classes in G^F_{un}. Let $\mathcal{N}$ be the set of all pairs $(\mathbf{c}, [\mathcal{F}])$ where $\mathbf{c}$ is a unipotent G^0-conjugacy class in G and $[\mathcal{F}]$ is an isomorphism class of an irreducible G^0-equivariant local system $\mathcal{F}$ on $\mathbf{c}$. Define $F : \mathcal{N} \to \mathcal{N}$ by $F(\mathbf{c}, [\mathcal{F}]) = (F(\mathbf{c}), [F_!\mathcal{F}])$. The fixed point set $\mathcal{N}^F$ is the set of all $(\mathbf{c}, [\mathcal{F}]) \in \mathcal{N}$ such that $F(\mathbf{c}) = \mathbf{c}$ and $F^*\mathcal{F} \cong \mathcal{F}$. For any $(\mathbf{c}, [\mathcal{F}]) \in \mathcal{N}^F$ we choose a local system $\mathcal{F} \in [\mathcal{F}]$ and an isomorphism $\phi_{\mathcal{F}} : F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. The function $\chi_{\mathcal{F},\phi_{\mathcal{F}}} : \mathbf{c}^F \to \bar{\mathbf{Q}}_l$ will be regarded as a function $G^{0F}_{un} \to \bar{\mathbf{Q}}_l$, equal to zero on $G^{0F}_{un} - \mathbf{c}^F$. Using Lemma 19.7, we see that

(a) *for any F-stable unipotent G^0-conjugacy class $\mathbf{c}'$ in G, the functions $\chi_{\mathcal{F}',\phi_{\mathcal{F}'}}$ with $(\mathbf{c}', [\mathcal{F}']) \in \mathcal{N}^F$ form a basis for the vector space of functions in $\mathcal{V}$ that are zero on $G^F_{un} - \mathbf{c}'^F$.*

From (a) we deduce

(b) *the functions $\chi_{\mathcal{F},\phi_{\mathcal{F}}}$ with $(\mathbf{c}, [\mathcal{F}]) \in \mathcal{N}^F$ form a basis of the vector space $\mathcal{V}$.*

For $(\mathbf{c}, [\mathcal{F}]) \in \mathcal{N}^F$ let $\mathcal{F}^\sharp = IC(\bar{\mathbf{c}}, \mathcal{F})$. Now $\phi_{\mathcal{F}}$ induces an isomorphism $\phi^\sharp_{\mathcal{F}} : F^*\mathcal{F}^\sharp \xrightarrow{\sim} \mathcal{F}^\sharp$ in $\mathcal{D}(\bar{\mathbf{c}})$. Hence $\chi_{\mathcal{F}^\sharp,\phi^\sharp_{\mathcal{F}}} : \bar{\mathbf{c}}^F \to \bar{\mathbf{Q}}_l$ is well defined. We regard $\chi_{\mathcal{F}^\sharp,\phi^\sharp_{\mathcal{F}}}$ as a function $G^F_{un} \to \bar{\mathbf{Q}}_l$, equal to zero on $G^F_{un} - \bar{\mathbf{c}}^F$. This function is constant on G^{0F}-conjugacy classes. Hence it is of the form $\sum_{\mathbf{c}'} c_{\mathbf{c}'} f_{\mathbf{c}'}$ where $\mathbf{c}'$ runs over the unipotent G^0-conjugacy classes in G such that $F(\mathbf{c}') = \mathbf{c}', \mathbf{c}' \subset \bar{\mathbf{c}}$, $c_{\mathbf{c}'} \in \bar{\mathbf{Q}}_l$ and $f_{\mathbf{c}'} \in \mathcal{V}$ is zero on $G^F - \mathbf{c}'^F$. For such $\mathbf{c}'$, $f_{\mathbf{c}'}$ is a linear combination of functions $\chi_{\mathcal{F}',\phi_{\mathcal{F}'}}$ with $\mathcal{F}'$ such that $(\mathbf{c}', [\mathcal{F}']) \in \mathcal{N}^F$ (see (a)). From the definitions we have $f_{\mathbf{c}} = \chi_{\mathcal{F},\phi_{\mathcal{F}}}$. We see that

$$\chi_{\mathcal{F}^\sharp,\phi^\sharp_{\mathcal{F}}} = \sum_{(\mathbf{c}', [\mathcal{F}']) \in \mathcal{N}^F} c_{[\mathcal{F}],[\mathcal{F}']} \chi_{\mathcal{F}',\phi_{\mathcal{F}'}}$$

where $c_{[\mathcal{F}],[\mathcal{F}']} \in \bar{\mathbf{Q}}_l$ are uniquely determined and equal to zero unless $\mathbf{c}' \subset \bar{\mathbf{c}}$; moreover, if $\mathbf{c}' = \mathbf{c}$ then $c_{[\mathcal{F}],[\mathcal{F}']} = \delta_{[\mathcal{F}],[\mathcal{F}']}$. Thus, the functions $\chi_{\mathcal{F}^\sharp,\phi^\sharp_{\mathcal{F}}}$ are related to the functions $\chi_{\mathcal{F},\phi_{\mathcal{F}}}$ by an upper triangular matrix with all diagonal entries equal to 1. Hence (b) implies:

(c) *the functions* $\chi_{\mathcal{F}^\sharp,\phi^\sharp_{\mathcal{F}}}$ *with* $(\mathbf{c},[\mathcal{F}]) \in \mathcal{N}^F$ *form a basis of the vector space* $\mathcal{V}$.

21.10. Let $\mathcal{Y}$ be the set of triples $(L,\mathfrak{c},[\mathfrak{f}])$ where L is the Levi of some parabolic of G^0, $\mathfrak{c}$ is a unipotent L-conjugacy class of $N_G L$ with $\mathfrak{c} \subset N^\bullet_G L$ and $[\mathfrak{f}]$ is the isomorphism class of an irreducible cuspidal local sytem $\mathfrak{f}$ on $\mathfrak{c}$ (relative to $N_G L$). Let $G^0\backslash\mathcal{Y}$ be the set of orbits of the natural G^0-action on $\mathcal{Y}$ given by conjugation all factors. Define $F : \mathcal{Y} \to \mathcal{Y}$ by $F(L,\mathfrak{c},[\mathfrak{f}]) = (F(L),F(\mathfrak{c}),[F_!\mathfrak{f}])$; we have $F(gy) = F(g)F(y)$ for all $g \in G^0$, $y \in \mathcal{Y}$. Hence F induces a bijection $F : G^0\backslash\mathcal{Y} \to G^0\backslash\mathcal{Y}$. Putting together the generalized Springer correspondences 11.10(a) for the various connected components of G that contain unipotent elements we obtain a canonical surjective map $\mathcal{N} \to G^0\backslash\mathcal{Y}$. From the definitions we see that this map is compatible with the F-actions, hence it restricts to a surjective map $\mathcal{N}^F \to (G^0\backslash\mathcal{Y})^F$ whose fibres form a partition of $\mathcal{N}^F$ into subsets $\mathcal{N}^F_\eta$ indexed by the F-stable G^0-orbits η on $\mathcal{Y}$. Using 21.9(c), we see that

(a) $\mathcal{V} = \oplus_\eta \mathcal{V}_\eta$

(η as above) where $\mathcal{V}_\eta$ is the subspace of $\mathcal{V}$ with basis formed by the functions $\chi_{\mathcal{F}^\sharp,\phi^\sharp_{\mathcal{F}}}$ with $(\mathbf{c},[\mathcal{F}]) \in \mathcal{N}^F_\eta$. Since η is a homogeneous space for the connected group G^0, it contains some F-fixed point $(L,\mathfrak{c},[\mathfrak{f}])$. We have $F(L) = L, F(\mathfrak{c}) = \mathfrak{c}$. We choose $\mathfrak{f} \in [\mathfrak{f}]$. We have $F^*\mathfrak{f} \cong \mathfrak{f}$; we choose an isomorphism $\epsilon_1 : F^*\mathfrak{f} \xrightarrow{\sim} \mathfrak{f}$. Let S be the stratum of $N_G L$ that contains $\mathfrak{c}$, let $\mathcal{E}$ be the inverse of $\mathcal{F}$ under $S \to \mathfrak{c}, g \mapsto g_u$ and let $\epsilon : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ be the isomorphism induced by ϵ_1. Then $(L,S,\mathcal{E},\epsilon)$ are as in 21.6.

We will apply 21.7(a) in our case. Restricting the functions in 21.7(a) to $\bar{Y}^{\omega F} = \bar{Y} \cap G^F_{un}$ ($\bar{Y}$ as in 21.6) we see that

(b) *the functions* $\chi_{{}^h\mathfrak{K},{}^h\phi}|_{\bar{Y}^{\omega F}}$ *($h \in I$) span the same vector space as the functions* $\chi_{A_j,\psi_j}|_{\bar{Y}^{\omega F}}$ *($j \in J$); moreover,* $|I| = |J|$.

From the definition of generalized Springer correspondence we see that the functions $\chi_{A_j,\psi_j}|_{\bar{Y}^{\omega F}}$, ($j \in J$), extended by 0 on $G^F_{un} - \bar{Y}^{\omega F}$, are up to non-zero scalars the same as the functions $\chi_{\mathcal{F}^\sharp,\phi^\sharp_{\mathcal{F}}}$ with $(\mathbf{c},[\mathcal{F}]) \in \mathcal{N}^F_\eta$. In particular, they form a basis of the vector space $\mathcal{V}_\eta$. Using now (b), we see that the functions $\chi_{{}^h\mathfrak{K},{}^h\phi}|_{\bar{Y}^{\omega F}}$, ($h \in I$), extended by 0 on $G^F_{un} - \bar{Y}^{\omega F}$ (or equivalently, the generalized Green functions $Q_{G,{}^hL,{}^h\mathfrak{c},{}^h\mathfrak{f},{}^h\epsilon_1} : G^F_{un} \to \bar{\mathbf{Q}}_l$, see below) form a basis of the vector space $\mathcal{V}_\eta$. Here ${}^h\mathfrak{c}$ is the set of unipotent elements in ${}^hS$, ${}^h\mathfrak{f} = {}^h\mathcal{E}|_{{}^h\mathfrak{c}}$ and ${}^h\epsilon_1$ is the restriction of ${}^h\epsilon$ to ${}^h\mathfrak{c}$. In our case

(c) *all triples in* Ξ^F *(see 21.7) are effective,*

since all elements of Γ are effective (see 21.6(e) and 21.8(a)). Letting now η vary and using (a), we obtain the following result.

Proposition 21.11. *The generalized Green functions* $Q_{G,L,\mathfrak{c},\mathfrak{f},\epsilon_1}$ *(where* $(L,\mathfrak{c},[\mathfrak{f}])$

runs through a set of representatives for the G^{0F}-orbits on $\mathcal{Y}^F$ and for each $(L,\mathfrak{c},[\mathfrak{f}])$ in this set we choose $\mathfrak{f}\in[\mathfrak{f}]$ and $\epsilon_1 : F^*\mathfrak{f}\xrightarrow{\sim}\mathfrak{f}$) form a basis of the $\bar{\mathbf{Q}}_l$-vector space $\mathcal{V}$ of functions $G^F_{un}\to\bar{\mathbf{Q}}_l$ that are constant on G^{0F}-conjugacy classes in G^F_{un}.

21.12. Define $F:\mathfrak{A}_G\to\mathfrak{A}_G$ by $F(L,\mathfrak{c})=(F(L),F(\mathfrak{c}))$. Let $(L,\mathfrak{c})\in\mathfrak{A}_G^F$. Then $\mathcal{C}_L(\mathfrak{c})$ is well defined (see 19.9).

(a) For any $n\in G^{0F}$ such that $nLn^{-1}=L, n\mathfrak{c}n^{-1}=\mathfrak{c}$ and any $f\in\mathcal{C}_L(\mathfrak{c}), g\in\mathfrak{c}^F$ we have $f(ngn^{-1})=f(g)$.

Indeed, we may assume that $f=\chi_{\mathfrak{f},\epsilon_1}$ where $\mathfrak{f}$ is an irreducible cuspidal local system on $\mathfrak{c}$ and $\epsilon_1 : F^*\mathfrak{f}\xrightarrow{\sim}\mathfrak{f}$ is an isomorphism. In that case the result follows from 21.10(c), since we have automatically $\mathrm{Ad}(n)^*\mathfrak{f}\cong\mathfrak{f}$ (see 11.7(a)).

We define a $\bar{\mathbf{Q}}_l$-linear map $\mathcal{C}_L(\mathfrak{c})\to\mathcal{V}, f\mapsto Q^f_{G,L,\mathfrak{c}}$ by the requirement that $Q^f_{G,L,\mathfrak{c}}=Q_{G,L,\mathfrak{c},\mathfrak{f},\epsilon_1}$ for any $f=\chi_{\mathfrak{f},\epsilon_1}$ as above. It is clear that this linear map is well defined. Let $J_{L,\mathfrak{c}}$ be the its image. Note that $J_{L,\mathfrak{c}}$ depends only on the G^{0F}-orbit of $(L,\mathfrak{c})$. We can reformulate Proposition 21.11 as follows.

(b) *For any $(L,\mathfrak{c})\in\mathfrak{A}_G^F$, the linear map $\mathcal{C}_L(\mathfrak{c})\to J_{L,\mathfrak{c}}$ is an isomorphism. We have a direct sum decomposition $\mathcal{V}=\oplus_{(L,\mathfrak{c})}J_{L,\mathfrak{c}}$ where $(L,\mathfrak{c})$ runs through a set of representatives for the G^{0F}-orbits on $\mathfrak{A}_G^F$.*

On $\oplus_{(L,\mathfrak{c})\in\mathfrak{A}_G^F}\mathcal{C}_L(\mathfrak{c})$ we have a linear G^F-action: an element $g\in G^F$ takes $f\in\mathcal{C}_L(\mathfrak{c})$ to $f'\in\mathcal{C}_{gLg^{-1}}(g\mathfrak{c}g^{-1})$ where $f'(h)=f(g^{-1}hg)$ for $h\in(g\mathfrak{c}g^{-1})^F$. This action restricts to a G^{0F}-action. Consider the linear map $\oplus_{(L,\mathfrak{c})\in\mathfrak{A}_G^F}\mathcal{C}_L(\mathfrak{c})\to\mathcal{V}$ whose restriction to any summand $\mathcal{C}_L(\mathfrak{c})$ is $f\mapsto Q^f_{G,L,\mathfrak{c}}$. Restricting this to the space of G^{0F}-invariants we obtain

(c) *an isomorphism* $(\oplus_{(L,\mathfrak{c})\in\mathfrak{A}_G^F}\mathcal{C}_L(\mathfrak{c}))^{G^{0F}}\xrightarrow{\sim}\mathcal{V}$.

This follows immediately from (b),(a).

21.13. Let $s\in G^F$ be a semisimple element and let $(L,\mathfrak{c})\in\mathfrak{A}^F_{Z_G(s)}$. Let $\mathcal{C}^s_L(\mathfrak{c})$ be the vector space $\mathcal{C}_L(\mathfrak{c})$ defined as in 19.9 with respect to $Z_G(s)$ instead of G. Let $'\mathcal{C}^s_L(\mathfrak{c})$ be the subspace of $\mathcal{C}^s_L(\mathfrak{c})$ consisting of all functions that are invariant under the natural action of $\{g\in Z_{G^0}(s)^F; gLg^{-1}=L, g\mathfrak{c}g^{-1}=\mathfrak{c}\}$. Replacing G by $Z_G(s)$ in 21.12(c) we obtain an isomorphism

$$(\oplus_{(L,\mathfrak{c})\in\mathfrak{A}^F_{Z_G(s)}}\mathcal{C}^s_L(\mathfrak{c}))^{Z_G(s)^{0F}}\xrightarrow{\sim}\mathcal{V}_s$$

where $\mathcal{V}_s$ is the vector space of functions $\{\text{unipotent elements in } Z_G(s)^F\}\to\bar{\mathbf{Q}}_l$ that are constant on $Z_G(s)^{0F}$-conjugacy classes. Taking now invariants for the natural action of $Z_{G^0}(s)^F$ (which contains $Z_G(s)^{0F}$ as a normal subgroup) we obtain an isomorphism

$$(\oplus_{(L,\mathfrak{c})\in\mathfrak{A}^F_{Z_G(s)}}\mathcal{C}^s_L(\mathfrak{c}))^{Z_{G^0}(s)^F}\xrightarrow{\sim}\mathcal{V}'_s$$

or, equivalently, an isomorphism

$$(*)\ (\oplus_{(L,\mathfrak{c})\in\mathfrak{A}^F_{Z_G(s)}}{}'\mathcal{C}^s_L(\mathfrak{c}))^{Z_{G^0}(s)^F}\xrightarrow{\sim}\mathcal{V}'_s$$

where $\mathcal{V}'_s$ is the vector space of functions $\{\text{unipotent elements in } Z_G(s)^F\}\to\bar{\mathbf{Q}}_l$ that are constant on $Z_{G^0}(s)^F$-conjugacy classes. We now take the direct sum of

these isomorphisms over all semisimple s in G^F and then take invariants for the natural action of G^{0F}. We obtain an isomorphism

(a) $$(\oplus_{(s,L,\mathfrak{c})\in\mathcal{X}}{}'\mathcal{C}_L^s(\mathfrak{c}))^{G^{0F}} \to VV$$

where $\mathcal{X}$ is the set of all triples $(s,L,\mathfrak{c})$ with $s\in G^F$ semisimple and $(L,\mathfrak{c})\in\mathfrak{A}^F_{Z_G(s)}$ and $\mathbf{V}$ is the vector space of functions $G^F \to \bar{\mathbf{Q}}_l$ that are constant on G^{0F}-conjugacy classes.

Let $\mathcal{X}^1$ be the set of all quadruples $(s,u,L,\mathfrak{c})$ where $s\in G^F$ is semisimple, $u\in G^F$ is unipotent, quasi-semisimple in $N_{Z_G(s)}L$, $(L,\mathfrak{c})\in\mathfrak{A}^F_{Z_G(s)}$, $\mathfrak{c}\subset uL$. Now G^{0F} acts naturally on $\mathcal{X}^1$ and the map $\mathcal{X}^1\to\mathcal{X}$, $(s,u,L,\mathfrak{c})\mapsto(s,L,\mathfrak{c})$ induces a bijection

(b) $G^{0F}\backslash\mathcal{X}^1 \xrightarrow{\sim} G^{0F}\backslash\mathcal{X}$

on the sets of G^{0F}-orbits. (We use the fact that, for fixed $(s,L,\mathfrak{c})\in\mathcal{X}$, the set of unipotent quasi-semisimple elements of $N_{Z_G(s)}L$ that are fixed by F and are contained in the component $\mathfrak{c}L$ of $N_{Z_G(s)}L$ is a single L^F-conjugacy class; this follows from 19.11.) Now $(g,L,\mathfrak{c})\mapsto(g_s,g_u,L,\mathfrak{c})$ is a bijection

(c) $\mathcal{R}'\xrightarrow{\sim}\mathcal{X}^1$.

(We use 1.4(c) and 21.2(a).) Combining (a),(b),(c) we obtain an isomorphism

(d) $$(\oplus_{(g,L,\mathfrak{c})\in\mathcal{R}'}{}'\mathcal{C}_L^{g_s}(\mathfrak{c}))^{G^{0F}} \xrightarrow{\sim} \mathbf{V}.$$

Assume that $(g,\underline{L},\mathbf{c})\in\mathcal{R}$ corresponds to $(g,L,\mathfrak{c})\in\mathcal{R}'$ under 21.4(c) and let $\langle g\rangle$ be the $\underline{L}^F$-conjugacy of g. From 19.15(b),(c) applied to $N_G\underline{L},\underline{L}$ instead of G,G^0 we have an isomorphism

(e) $$\mathcal{C}_{\underline{L},\langle g\rangle}(\mathbf{c}) \xrightarrow{\sim} \mathcal{C}_{H_{\underline{L}}(g)}(\mathfrak{c})$$

where $H_{\underline{L}}(g)=\{l\in\underline{L}; lg_sl^{-1}=g_s, lg_ul^{-1}\in Z_{\underline{L}}(g_s)^0g_u\}$.

Let $\mathcal{G}=\{y\in Z_{G^0}(g_s)^F; yLy^{-1}=L, y\mathfrak{c}y^{-1}=\mathfrak{c}\}$ (a group containing $H_{\underline{L}}(g)^F$ as a normal subgroup).

Let $\mathcal{G}'=\{y\in G^{0F}, y\underline{L}y^{-1}=\underline{L}, y\mathbf{c}y^{-1}=\mathbf{c}, y\langle g\rangle y^{-1}=\langle g\rangle\}$ (a group containing $\underline{L}^F$ as a normal subgroup).

Assume that $f\mapsto\bar{f}$ under (e). We show that the following two conditions are equivalent:

(i) $\bar{f}$ is invariant under the natural action of $\mathcal{G}$;

(ii) f is invariant under the natural action of $\mathcal{G}'$.

Assume that (i) holds. Let $y\in\mathcal{G}'$. Then $y=l'y'$ where $l'\in\underline{L}^F, y'\in\mathcal{G}$ (see 21.5(a)). We must show that $f(ylg_sul^{-1}y^{-1})=f(lg_sul^{-1})$ for $l\in\underline{L}^F, u\in\mathfrak{c}^F$ or that $f(l'y'lg_sul^{-1}y'^{-1}l'^{-1})=\bar{f}(u)$ or that $\bar{f}(y'uy'^{-1})=\bar{f}(u)$; this follows from $y'\in\mathcal{G}$.

Assume that (ii) holds. Let $y\in\mathcal{G}$. Then $y\in\mathcal{G}'$ (see 21.5(a)). We must show that $\bar{f}(yuy^{-1})=\bar{f}(u)$ for $u\in\mathfrak{c}^F$ or that $f(g_syuy^{-1})=f(g_su)$ or that $f(yg_suy^{-1})=f(g_su)$; this follows from $y\in\mathcal{G}'$.

From the equivalence of (i),(ii), we see that (e) restricts to an isomorphism

(f) $$'\mathcal{C}_{\underline{L},\langle g\rangle}(\mathbf{c}) \xrightarrow{\sim} {}'\mathcal{C}_L^{g_s}(\mathfrak{c})$$

where $'\mathcal{C}_{\underline{L},\langle g\rangle}(\mathbf{c})$ is the space of all functions $f \in \mathcal{C}_{\underline{L},\langle g\rangle}(\mathbf{c})$ that are invariant under the natural action of $\{y \in G^{0F}, y\underline{L}y^{-1} = \underline{L}, y\mathbf{c}y^{-1} = \mathbf{c}, y\langle g\rangle y^{-1} = \langle g\rangle\}$. Using (f), we deduce from (d) an isomorphism

$$(\oplus_{(g,\underline{L},\mathbf{c})\in\mathcal{R}}{}'\mathcal{C}_{\underline{L},\langle g\rangle}(\mathbf{c}))^{G^{0F}} \xrightarrow{\sim} \mathbf{V}$$

or equivalently an isomorphism

$$(\oplus_{(\underline{L},\mathbf{c})\in\mathfrak{A}_G^F} \oplus_\gamma {}'\mathcal{C}_{\underline{L},\gamma}(\mathbf{c}))^{G^{0F}} \to \mathbf{V}$$

where γ runs through the set of $\underline{L}^F$-orbits on $(\sigma_{N_G\underline{L}}\mathbf{c})^F$, or equivalently an isomorphism

$$(\oplus_{(\underline{L},\mathbf{c})\in\mathfrak{A}_G^F} \oplus_\gamma \mathcal{C}_{\underline{L},\gamma}(\mathbf{c}))^{G^{0F}} \xrightarrow{\sim} \mathbf{V}.$$

From the definitions we have canonically $\oplus_\gamma\mathcal{C}_{\underline{L},\gamma}(\mathbf{c}) = \mathcal{C}_{\underline{L}}(\mathbf{c})$ hence we obtain an isomorphism

(g) $$(\oplus_{(\underline{L},\mathbf{c})\in\mathfrak{A}_G^F}\mathcal{C}_{\underline{L}}(\mathbf{c}))^{G^{0F}} \xrightarrow{\sim} \mathbf{V}.$$

We define $F : \mathbf{A} \to \mathbf{A}$ by $F(\underline{L}, S) = (F(\underline{L}), F(S))$. There is a well defined surjective map $\mathfrak{A}_G^F \to \mathbf{A}^F$ given by $(\underline{L}, \mathbf{c}) \mapsto (\underline{L}, S)$ where S is defined by $\mathbf{c} \subset S$. Moreover, if $(\underline{L}, S) \in \mathbf{A}^F$ is given we have a natural isomorphism $\mathcal{C}_{\underline{L}}(S) \xrightarrow{\sim} \oplus_{\mathbf{c}}\mathcal{C}_{\underline{L}}(\mathbf{c})$ where $\mathbf{c}$ runs over the F-stable $\underline{L}$-conjugacy classes contained in S. (A special case of 19.10(b).) Introducing this in (g) we obtain an isomorphism

$$(\oplus_{(\underline{L},S)\in\mathbf{A}^F}\mathcal{C}_{\underline{L}}(S))^{G^{0F}} \xrightarrow{\sim} \mathbf{V}.$$

For each $(\underline{L}, S) \in \mathbf{A}^F$ we have a canonical direct sum decomposition

$$\mathcal{C}_{\underline{L}}(S) = \oplus_{[\mathcal{E}]}\mathcal{C}_{\underline{L}}^{[\mathcal{E}]}(S)$$

where $[\mathcal{E}]$ runs over the set of isomorphism classes of irreducible cuspidal local systems $\mathcal{E} \in \mathcal{S}(S)$ (relative to $N_G\underline{L}$) such that $F^*\mathcal{E} \cong \mathcal{E}$ and $\mathcal{C}_{\underline{L}}^{[\mathcal{E}]}(S)$ is the line spanned by $\chi_{\mathcal{E},\epsilon}$ where $\mathcal{E} \in [\mathcal{E}]$ and $\epsilon : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. (This follows from 19.8(a).) Hence we have an isomorphism

(h) $$(\oplus_{(\underline{L},S,[\mathcal{E}])\in\tilde{\mathbf{A}}^F}\mathcal{C}_{\underline{L}}^{[\mathcal{E}]}(S))^{G^{0F}} \xrightarrow{\sim} \mathbf{V}$$

where $\tilde{\mathbf{A}}^F$ is the set of triples $(\underline{L}, S, [\mathcal{E}])$ with $(\underline{L}, S) \in \mathbf{A}^F$ and $[\mathcal{E}]$ is as above. The left hand side of (h) is naturally a direct sum of subspaces $V_{(\underline{L},S,[\mathcal{E}])}$ indexed

by a set of representatives for the G^{0F}-orbits on $\tilde{\mathbf{A}}^F$ and $V_{(\underline{L},S,[\mathcal{E}])}$ is the space of vectors in the one dimensional vector space $\mathcal{C}_{\underline{L}}^{[\mathcal{E}]}(S)$ that are invariant under the natural action of the group $\{g \in G^{0F}; gLg^{-1} = L, gSg^{-1} = S, \mathrm{Ad}(g)^*\mathcal{E} \cong \mathcal{E}\}$. From the definitions we see that $V_{(\underline{L},S,[\mathcal{E}])}$ is 1-dimensional if $(\underline{L}, S, \mathcal{E})$ is effective and is 0 if $(\underline{L}, S, \mathcal{E})$ is not effective.

Thus the left hand side of (h) has dimension equal to the number of G^{0F}-orbits on the set of effective triples in $\tilde{\mathbf{A}}^F$. Hence this number is equal to the dimension of the right hand side that is, to the number of G^{0F}-conjugacy classes in G^F.

Theorem 21.14. *Let $\mathcal{A}$ be a set of representatives for the G^{0F}-orbits on the set of effective triples in $\tilde{\mathbf{A}}^F$. For each $(\underline{L}, S, [\mathcal{E}]) \in \mathcal{A}$ we choose $\mathcal{E} \in [\mathcal{E}]$ and an isomorphism $\epsilon : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. To $\underline{L}, S, \mathcal{E}, \epsilon$ we associate $\mathfrak{K} \in \mathcal{D}(G)$ and $\phi : F^*\mathfrak{K} \xrightarrow{\sim} \mathfrak{K}$ as in 21.6 (with $\underline{L}$ instead of L). The functions $\chi_{\mathfrak{K},\phi}$ (one for each $(\underline{L}, S, [\mathcal{E}]) \in \mathcal{A}$) form a $\bar{\mathbf{Q}}_l$-basis of the vector space $\mathbf{V}$ of functions $G^F \to \bar{\mathbf{Q}}_l$ that are constant on G^{0F}-conjugacy classes.*

Let $\mathbf{V}'$ be the subspace of $\mathbf{V}$ spanned by the functions $\chi_{\mathfrak{K},\phi}$ in the theorem. By the last paragraph in 21.13, it is enough to show that $\mathbf{V}' = \mathbf{V}$. This will be done in 21.17. Note that in the definition of $\mathbf{V}'$ we may include the functions $\chi_{\mathfrak{K},\phi}$ corresponding to non-effective triples in $\tilde{\mathbf{A}}$ (these functions are identically 0 by 21.6(f)).

21.15. Let $(L, S) \in \mathbf{A}^F$. We define a linear function $\Psi : \mathcal{C}_L(S) \to \mathbf{V}$ by the requirement that for any irreducible cuspidal local system $\mathcal{E} \in \mathcal{S}(S)$ and any $\epsilon : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ we have $\Psi(\chi_{\mathcal{E},\epsilon}) = \chi_{\mathfrak{K},\phi}$ where $\mathfrak{K}, \phi$ are defined as in 21.6. From Theorem 16.14 we see that for any $f \in \mathcal{C}_L(S)$ and any $y \in G^F$ we have

(a) $$\Psi(f)(y) = \sum_{x \in G^{0F}, \mathbf{d}; x^{-1}y_s x \in S_s} |L_x^F||Z_G(y_s)^{0F}|^{-1}|L^F|^{-1} Q_{L_x, Z_G(y_s), \mathbf{d}}^{f_x^{\mathbf{d}}}(y_u)$$

where $\mathbf{d}$ runs over the set of F-stable $Z_G(y_s)^0$-orbits contained in
$\{v \in Z_G(y_s); v \text{ unipotent}, x^{-1}y_s vx \in S\}$,
$L_x = Z_{xLx^{-1}}(y_s)^0$ (a Levi of some parabolic of $Z_G(y_s)^0$), and $f_x^{\mathbf{d}} \in \mathcal{C}_{L_x}(\mathbf{d})$ is defined by $f_x^{\mathbf{d}}(v) = f(x^{-1}y_s vx)$. (Notation of 21.12.)

21.16. Assume that we are given an F-stable L-conjugacy class $\mathbf{c}$ in S and a quasi-semisimple element $g \in N_G L$ such that $F(g) = g$ and such that
$\mathfrak{c} = \{u \in Z_L(s)^0 g_u; u \text{ unipotent }, su \in \mathbf{c}\} \neq \emptyset$.
Here $s = g_s$. We assume that $\mathbf{c}$ is cuspidal (relative to $N_G L$). Then $\mathfrak{c}$ is a single $Z_L(g_s)^0$-conjugacy class (see 17.13). Assume that f in 21.15 satisfies:
(i) $f|_{S^F - \mathbf{c}^F} = 0$,
(ii) $h \in \mathbf{c}^F, f(h) \neq 0 \implies h = lsul^{-1}$ for some $l \in L^F$ and some $u \in \mathfrak{c}^F$.
Consider the function $\tilde{f} \in \mathbf{V}$ given on $y \in G^F$ by
(a) $\tilde{f}(y) = 0$ if y_s is not G^{0F}-conjugate to s;

$\tilde f(y)=|Z_G(s)^{0F}|^{-1}\sum_{z\in Z_{G^0}(s)^F}Q^h_{L_1,Z_G(s),\mathfrak{c}}(z^{-1}y_uz)$ if $y_s=s$;
here $L_1=Z_L(y_s)^0$ and $h:\mathfrak{c}^F\to\bar{\mathbf{Q}}_l$ is given by $h(v)=f(y_sv)$. We show that
(b) $\Psi(f)=|L_1^F||Z_L(s)^F|^{-1}\tilde f$.
Consider the sum over $x,\mathbf{d}$ in 21.15(a) for our f. If $f_x^{\mathbf{d}}$ is not identically 0 then $f(x^{-1}y_svx)\ne 0$ for some $v\in\mathbf{d}^F$. Hence there exist $l\in L^F$, $u\in\mathfrak{c}^F$ such that $x^{-1}y_svx=lsul^{-1}$ where $v\in\mathbf{d}^F$. Thus, $xls(xl)^{-1}=y_s, xlu(xl)^{-1}=v$. We see that $\Psi(f)(y)=0$ if y_s is not G^{0F}-conjugate to s. Assume now that $y_s=s$. Setting $z=xl$ we have $z\in Z_{G^0}(s)^F, zuz^{-1}=v$. Hence $\mathbf{d}=z\mathfrak{c}z^{-1}$. We see that

$$\Psi(f)(y)=\sum_{z\in Z_{G^0}(s)^F,l\in L^F}|L_z^F||Z_G(s)^{0F}|^{-1}|L^F|^{-1}|Z_L(s)^F|^{-1}Q^{f_z^{z\mathfrak{c}z^{-1}}}_{L_z,Z_G(s),z\mathfrak{c}z^{-1}}(y_u)$$

$$=|L_1^F||Z_G(s)^{0F}|^{-1}|Z_L(s)^F|^{-1}\sum_{z\in Z_{G^0}(s)^F}Q^{f_1^{\mathfrak{c}}}_{L_1,Z_G(s),\mathfrak{c}}(z^{-1}y_uz),$$

as desired.

21.17. From the definitions we see that $\mathbf{V}'$ is the subspace of $\mathbf{V}$ spanned by the functions $\Psi(f)$ for various $(L,S)\in\mathbf{A}^F$ and $f\in\mathcal{C}_L(S)$. Since the vector space $\mathcal{C}_L(S)$ is spanned by functions as in 21.16(i),(ii) (see 19.10(b),19.14(c)) we see that any function $\tilde f$ as in 21.16(a) is contained in $\mathbf{V}'$. In formula 21.16(a) defining $\tilde f$ we may take $(g,L_1,\mathfrak{c})$ to be any triple in $\mathcal{R}'$ and h to be any function in $\mathcal{C}_{L_1}(\mathfrak{c})$ that is invariant under the natural action of $\{g'\in Z_{G^0}(s)^F;g'L_1g'^{-1}=L_1,g'\mathfrak{c}g'^{-1}=\mathfrak{c}\}$. (This follows from the bijection $\mathcal{R}\leftrightarrow\mathcal{R}'$ in 21.4(c), the isomorphism 19.15(c) and the equivalence of (i),(ii) in 21.13.) But such $\tilde f$ span the vector space $\mathbf{V}_s$ of functions in $\mathbf{V}$ that vanish at elements whose semisimple parts are not G^{0F}-conjugate to s. ($\mathbf{V}_s$ may be identified with $\mathcal{V}'_s$ in 21.13 and we may use the isomorphism 21.13($*$).) We see that $\mathbf{V}_s\subset\mathbf{V}'$. Since $\mathbf{V}$ is the sum of its subspaces $\mathbf{V}_s$ for various s, we see that $\mathbf{V}\subset\mathbf{V}'$. Hence $\mathbf{V}=\mathbf{V}'$. Theorem 21.14 is proved.

21.18. Let J be the set of all triples $(L,S,[\mathcal{E}])$ where $(L,S)\in\mathbf{A}$ and $[\mathcal{E}]$ is the isomorphism class of an irreducible cuspidal local system $\mathcal{E}\in\mathcal{S}(S)$. The group G^0 acts on J by $g:(L,S,[\mathcal{E}])\mapsto(\mathrm{Ad}(g)L,\mathrm{Ad}(g)S,[\mathrm{Ad}(g^{-1})^*\mathcal{E}])$; let $G^0\backslash J$ be the set of orbits.

Let $\mathfrak{A}(G)$ be the subcategory of $\mathcal{D}(G)$ whose objects are the complexes X on G such that $X[d]$ is a (simple) admissible perverse sheaf on G with support of dimension d (see 6.7).

Let $\underline{\mathfrak{A}}(G)$ be the set of isomorphism classes of objects in $\mathfrak{A}(G)$. We define a map

$$j:\underline{\mathfrak{A}}(G)\to G^0\backslash J$$

as follows. Let $A\in\mathfrak{A}(G)$. By definition there exists $(L,S,\mathcal{E})$ as above such that A is isomorphic to a direct summand of $IC(\bar Y,\pi_!\tilde{\mathcal{E}})$ (extended by 0 on $G-\bar Y$), with $\pi,\tilde{\mathcal{E}},\bar Y$ as in 5.6. Then j takes the isomorphism class of A to the G^0-orbit of $(L,S,[\mathcal{E}])$. To show that this is well defined we must show that, if $(L',S',\mathcal{E}')$

is another triple like $(L,S,\mathcal{E})$ such that A is isomorphic to a direct summand of $IC(\bar{Y}', \pi'_! \tilde{\mathcal{E}}')$ (extended by 0 on $G - \bar{Y}'$), with $\pi', \tilde{\mathcal{E}}', \bar{Y}'$ defined as in 5.6 in terms of $L', S', \mathcal{E}'$ instead of $L, S, \mathcal{E}$, then $(L, S, [\mathcal{E}])$, $(L', S', [\mathcal{E}'])$ are in the same G^0-orbit. Now A is an intersection cohomology complex supported by $\bar{Y}$ and also by $\bar{Y}'$. It follows that $\bar{Y} = \bar{Y}'$ hence $Y_{L,S} = Y_{L',S'}$. Using 3.12(b) we deduce that $(L,S), (L',S')$ are in the same G^0-orbit. Hence we may assume that $L = L', S = S'$. Then $\mathcal{E}, \mathcal{E}' \in \mathcal{S}(S)$ and there exists an irreducible local system on $\bar{Y}$ which is a direct summand of both $\pi_! \tilde{\mathcal{E}}$ and $\pi_! \tilde{\mathcal{E}}'$. Thus, $\mathrm{Hom}(\pi_! \tilde{\mathcal{E}}, \pi_! \tilde{\mathcal{E}}') \neq 0$. We now repeat an argument in 7.10 (and use notation there):

$$
\begin{aligned}
&\mathrm{Hom}(\pi_! \tilde{\mathcal{E}}, \pi_! \tilde{\mathcal{E}}') = \mathrm{Hom}(\pi^* \pi_! \tilde{\mathcal{E}}, \tilde{\mathcal{E}}') = \oplus_{w \in \mathcal{W}_S} \mathrm{Hom}(f_w^* \tilde{\mathcal{E}}, \tilde{\mathcal{E}}') \\
&= \oplus_{w \in \mathcal{W}_S} \mathrm{Hom}(a^* f_w^* \tilde{\mathcal{E}}, a^* \tilde{\mathcal{E}}') = \oplus_{w \in \mathcal{W}_S} \mathrm{Hom}(\hat{f}_w^* a^* \tilde{\mathcal{E}}, a^* \tilde{\mathcal{E}}') \\
&= \oplus_{w \in \mathcal{W}_S} \mathrm{Hom}(\hat{f}_w^* b^* \mathcal{E}, b^* \mathcal{E}') = \oplus_{w \in \mathcal{W}_S} \mathrm{Hom}(b^* \mathrm{Ad}(n_w)^* \mathcal{E}, b^* \mathcal{E}') \\
&= \oplus_{w \in \mathcal{W}_S} \mathrm{Hom}(\mathrm{Ad}(n_w)^* \mathcal{E}, \mathcal{E}').
\end{aligned}
$$

We see that $\oplus_{w \in \mathcal{W}_S} \mathrm{Hom}(\mathrm{Ad}(n_w)^* \mathcal{E}, \mathcal{E}') \neq 0$. Hence $\mathrm{Hom}(\mathrm{Ad}(n_w)^* \mathcal{E}, \mathcal{E}') \neq 0$ for some $w \in \mathcal{W}_S$, so that $\mathrm{Ad}(n_w)^* \mathcal{E} \cong \mathcal{E}'$. Thus, $(L, S, [\mathcal{E}])$ is in the same G^0-orbit as $(L, S, [\mathcal{E}'])$, as required.

21.19. From the definitions we see that the map j in 21.18 is compatible with the maps $F : \underline{\mathfrak{A}}(G) \longrightarrow \underline{\mathfrak{A}}(G)$, $F : G^0 \backslash J \longrightarrow G^0 \backslash J$ defined by $A \mapsto F_! A$, $(L, S, [\mathcal{E}]) \mapsto (F(L), F(S), [F_! \mathcal{E}])$. Hence it induces a map $j_0 : \underline{\mathfrak{A}}(G)^F \longrightarrow (G^0 \backslash J)^F$ on the fixed point sets of F.

21.20. From Theorem 21.14 we see that $\mathbf{V} = \oplus_\Xi \mathbf{V}^\Xi$ where Ξ runs over the set of F-stable G^0-orbits in J and $\mathbf{V}^\Xi$ is the subspace of $\mathbf{V}$ with basis given by the characteristic functions of F-stable effective triples $(L, S, [\mathcal{E}]) \in \Xi$ (up to the action of G^{0F}). Using now 21.7(a) we see that another basis of $\mathbf{V}^\Xi$ is given by the characteristic functions of objects in $j_0^{-1}(\Xi)$. (Either of these bases is defined only up to multiplication of any of its members by a non-zero scalar.) Here we have used the fact that Ξ contains at least one F-stable triple $(L, S, [\mathcal{E}])$ which follows from Lang's theorem for G^0 since Ξ is a homogeneous G^0-space. Since $\underline{\mathfrak{A}}(G)^F = \sqcup_\Xi j_0^{-1}(\Xi)$, we see that the following result holds.

Theorem 21.21. *Let $\mathcal{A}'$ be a set of representatives for the isomorphism classes of objects $A \in \mathfrak{A}$ such that $F^* A \cong A$. For each $A \in \mathcal{A}'$ we choose an isomorphism $\alpha : F^* A \xrightarrow{\sim} A$. The functions $\chi_{A,\alpha}$ (one for each $A \in \mathcal{A}'$) form a $\bar{\mathbf{Q}}_l$-basis of the vector space $\mathbf{V}$ of functions $G^F \longrightarrow \bar{\mathbf{Q}}_l$ that are constant on G^{0F}-conjugacy classes.*

22. Twisted induction of class functions

22.1. This section gives an application of Theorem 21.14 to the construction of a "twisted induction" map (see 22.3) from certain functions on a subgroup of G^F to functions on G^F.

Lemma 22.2. *Let L be a Levi of a parabolic of G^0 and let L' be a Levi of a parabolic of L. Let δ' be a connected component of $N_{N_GL}L'$ and let δ be the connected component of N_GL that contains δ'. Assume that $\delta' \subset N^\bullet_{N_GL}(L')$ and $\delta \subset N^\bullet_G L$. Then $\delta' \subset N^\bullet_G(L')$.*

Since $\delta \subset N^\bullet_G L$, there exists a parabolic P of G^0 such that L is a Levi of P and $\delta \subset N_G P$. Since $\delta' \subset N^\bullet_{N_GL}(L')$, there exists a parabolic Q of L such that L' is a Levi of Q and $\delta' \subset N_{N_GL}Q$. Then $P' = QU_P$ is a parabolic of G^0 such that L' is a Levi of P'. If $g \in \delta'$ then $gQg^{-1} = Q$ and $gU_Pg^{-1} = U_P$ (since $g \in \delta \subset N_G P$) hence $gP'g^{-1} = P'$. Thus $\delta' \subset N_G P'$. We see that $\delta' \subset N^\bullet_G(L')$, as required.

22.3. Let L be a Levi of a parabolic of G^0 and let δ be a connected component of N_GL contained in $N^\bullet_G L$. We assume that $F(L) = L, F(\delta) = \delta$. Let D be the connected component of G that contains δ. Let $\mathbf{V}_L(\delta)$ (resp. $\mathbf{V}_{G^0}(D)$) be the set of all functions $\delta^F \to \bar{\mathbf{Q}}_l$ (resp. $D^F \to \bar{\mathbf{Q}}_l$) that are constant on L^F-conjugacy classes in δ (resp. on G^{0F}-conjugacy classes in D).

There is a unique $\bar{\mathbf{Q}}_l$-linear map

$$R^D_\delta : \mathbf{V}_L(\delta) \to \mathbf{V}_{G^0}(D)$$

such that the following holds.

Let L' be a Levi of a parabolic of L with $F(L') = L'$, let S' be an isolated stratum of $N_{N_GL}(L') = N_GL \cap N_GL'$ with $F(S') = S', S' \subset \delta$, $S' \subset N^\bullet_{N_GL}(L')$, let $\mathcal{E}'$ be an irreducible cuspidal local system in $\mathcal{S}(S')$ and let $\epsilon' : F^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}'$ be an isomorphism. Define $\mathfrak{K}' \in \mathcal{D}(N_GL)$, $\phi' : F^*\mathfrak{K}' \xrightarrow{\sim} \mathfrak{K}'$ in terms of $N_GL, L', S', \mathcal{E}', \epsilon'$ and $\mathfrak{K}'' \in \mathcal{D}(G)$, $\phi'' : F^*\mathfrak{K}'' \xrightarrow{\sim} \mathfrak{K}''$ in terms of $G, L', S', \mathcal{E}', \epsilon'$ in the same way as $\mathfrak{K} \in \mathcal{D}(G), \phi : F^*\mathfrak{K} \xrightarrow{\sim} \mathfrak{K}$ were defined in 21.6 in terms of $G, L, S, \mathcal{E}, \epsilon$. (Note that $\mathfrak{K}''$ is well defined since $S' \subset N^\bullet_G L'$, by Lemma 22.2.) Then

$$R^D_\delta(\chi_{\mathfrak{K}',\phi'}|_{\delta^F}) = \chi_{\mathfrak{K}'',\phi''}|_{D^F}.$$

To see that this definition is correct we use the fact that the functions $\chi_{\mathfrak{K}',\phi'}|_{\delta^F}$ as above (with ϵ' chosen for each $L', S', \mathcal{E}'$ given up to L^F-conjugacy) provide a basis for $\mathbf{V}_L(\delta)$ (which follows from Theorem 21.14 for N_GL instead of G); note that the choice of ϵ' is immaterial since the same choice is made in the definition of $\chi_{\mathfrak{K}'',\phi''}|_{D^F}$.

22.4. For any G^{0F}-conjugacy class c of semisimple elements in G^F let $\mathbf{V}_{G^0,c}(D)$ be the vector space consisting of all functions in $\mathbf{V}_{G^0}(D)$ that vanish on elements $g \in D^F$ with $g_s \notin c$. We have a direct sum decomposition

(a) $\mathbf{V}_{G^0}(D) = \oplus_c \mathbf{V}_{G^0,c}(D)$

where c runs over the semisimple G^{0F}-conjugacy classes in G^F. Similarly we have a direct sum decomposition

(b) $\mathbf{V}_L(\delta) = \oplus_{c'} \mathbf{V}_{L,c'}(\delta)$

where c' runs over the semisimple L^F-conjugacy classes in $(N_GL)^F$. The next result shows that R^D_δ is compatible with the direct sum decompositions (a),(b).

Proposition 22.5. *Let c' be any semisimple L^F-conjugacy class in $(N_GL)^F$ and let c be the semisimple G^{0F}-conjugacy class in G^F such that $c' \subset c$. Then $R_\delta^D(\mathbf{V}_{L,c'}(\delta)) \subset \mathbf{V}_{G^0,c}(D)$.*

Let L', S' be as in 22.3. As in 21.15 we define linear maps $\Psi' : \mathcal{C}_{L'}(S') \to \mathbf{V}_L(\delta)$, $\Psi'' : \mathcal{C}_{L'}(S') \to \mathbf{V}_{G^0}(D)$ by the requirement that for any $\mathcal{E}', \epsilon'$ as in 22.3 we have $\Psi'(\chi_{\mathcal{E}',\epsilon'}) = \chi_{\mathfrak{K}',\phi'}|_{\delta^F}$, $\Psi''(\chi_{\mathcal{E}',\epsilon'}) = \chi_{\mathfrak{K}'',\phi''}|_{D^F}$ (notation of 22.3). Clearly,

(a) $R_\delta^D(\Psi'(f)) = \Psi''(f)$

for $f = \chi_{\mathcal{E}',\epsilon'}$ hence also for any $f \in \mathcal{C}_{L'}(S')$. Assume now that $\mathbf{c}$ is an F-stable L'-conjugacy class in S' and that g is a quasi-semisimple element in $N_{N_GL}(L')$ such that $g \in \delta, F(g) = g$ and such that $\mathfrak{c} = \{u \in Z_{L'}(s)^0 g_u; u \text{ unipotent }, su \in \mathbf{c}\} \neq \emptyset$ (with $s = g_s$). Assume that $\mathbf{c}$ is cuspidal (relative to $N_{N_GL}(L')$) so that $\mathfrak{c}$ is a single $Z_{L'}(s)^0$-conjugacy class. Assume that $f \in \mathcal{C}_{L'}(S')$ satisfies

(i) $f|_{S'^F - \mathbf{c}^F} = 0$,

(ii) $h \in \mathbf{c}^F, f(h) \neq 0 \implies h = lsul^{-1}$ for some $l \in L'^F$ and some $u \in \mathfrak{c}^F$.

Consider the functions $\tilde{f} \in \mathbf{V}_L(\delta), \tilde{f}'' \in \mathbf{V}_{G^0}(D)$ defined by

$\tilde{f}(y) = 0$ if $y \in \delta^F$, y_s is not L^F-conjugate to s;

$\tilde{f}(y) = |Z_L(s)^{0F}|^{-1} \sum_{z \in Z_L(s)^F} Q^h_{L'_1, Z_{N_GL}(s), \mathfrak{c}}(z^{-1}y_u z)$ if $y \in \delta^F, y_s = s$;

$\tilde{f}''(y) = 0$ if $y \in D^F$, y_s is not G^{0F}-conjugate to s;

$\tilde{f}''(y) = |Z_G(s)^{0F}|^{-1} \sum_{z \in Z_{G^0}(s)^F} Q^h_{L'_1, Z_G(s), \mathfrak{c}}(z^{-1}y_u z)$ if $y \in D^F, y_s = s$;

here $L'_1 = Z_{L'}(s)^0$ and $h : \mathfrak{c}^F \to \bar{\mathbf{Q}}_l$ is given by $h(v) = f(y_s v)$. Using 21.16(f) for G and for N_GL and (a) we see that

(b) $R_\delta^D(\tilde{f}) = \tilde{f}''$.

As in 21.17, here we may assume that $(g, L'_1, \mathfrak{c})$ used in the definition of $\tilde{f}, \tilde{f}''$ is any triple in $\mathcal{R}'$ (for N_GL instead of G) with $g \in \delta$ and h is any function in $\mathcal{C}_{L'_1}(\mathfrak{c})$ that is invariant under the natural action of $\{g' \in Z_L(s)^F; g'L'_1g'^{-1} = L'_1, g'\mathfrak{c}g'^{-1} = \mathfrak{c}\}$. Since the functions $\tilde{f}$ as above span the vector space $\mathbf{V}_{L,c'}(\delta)$ where c' is the L^F-conjugacy class of g_s and the corresponding functions $\tilde{f}''$ are contained in the corresponding $\mathbf{V}_{G^0,c}(D)$ we see that the Proposition follows from (b).

22.6. Let $L' \subset L \subset G^0, \delta' \subset \delta$ be as in Lemma 22.2 and let D be the connected component of G that contains δ. Then $R^\delta_{\delta'}, R^D_\delta, R^D_{\delta'}$ are well defined (the last one, by Lemma 22.2). From the definitions we see that the transitivity property

(a) $R^D_\delta \circ R^\delta_{\delta'} = R^D_{\delta'}$

holds.

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